Fluctuating nematic elastomer membranes

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We study the flat phase of nematic elastomer membranes with rotational symmetry spontaneously broken by an in-plane nematic order. Such a state is characterized by a vanishing elastic modulus for simple shear and soft transverse phonons. At harmonic level, the in-plane orientational (nematic) order is stable to thermal fluctuations that lead to short-range in-plane translational (phonon) correlations. To treat thermal fluctuations and relevant elastic nonlinearities, we introduce two generalizations of two-dimensional membranes in a three-dimensional space to arbitrary D-dimensional membranes embedded in a d-dimensional space and analyze their anomalous elasticities in an expansion about $D=4$. We find a stable fixed point that controls long-scale properties of nematic elastomer membranes. It is characterized by singular in-plane elastic moduli that vanish as a power law $\eta_\lambda=4-D$ of a relevant inverse length scale (e.g., wave vector) and a finite bending rigidity. Our predictions are asymptotically exact near four dimensions.

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I. INTRODUCTION

The ubiquity and importance of membrane realizations in nature, such as cellular walls, and in laboratories, such as self-assembled bilayers of lipid amphiphils, have stimulated considerable scientific activities [1]. These nearly tensionless [2] lipid sheets are highly flexible with elastic moduli often comparable to thermal energies. Consequently, on long length scales, their conformational properties are strongly affected by thermal fluctuations. This, together with early successes in understanding a variety of puzzling phenomena (such as red blood cells flicker [3], biconcave shape of eurythrocites [4], and period of lyotropics) in terms of continuum models of fluctuating elastic sheets, has attracted the attention of the physics community. Consequently, significant progress has been made in understanding the statistical mechanics of fluctuating membranes [1].

It is by now well appreciated that the nature of a membrane's in-plane order, with three (heretofore studied) universality classes, the isotropic, hexatic, and solid (i.e., tethered or polymerized), crucially affects its conformational properties. The most striking effect of in-plane orders is the stabilization in solid membranes of a ''flat'' phase [5], with a long-range orientational order in the local membrane normals [6], that is favored at low temperature over the entropically preferred high-temperature crumpled state. Therefore, in marked contrast to liquid membranes and one-dimensional polymer analogs, which are always crumpled (beyond a persistence length) [7], tethered membranes, despite being two dimensional [8] are predicted [5] to undergo a thermodynamically sharp crumpled-to-flat phase transition [9,10]. The ordering is made possible by a subtle interplay of thermal fluctuations with nonlinear membrane elasticity, which at long scales infinitely enhances a membrane's bending rigidity, thereby stabilizing the orientational order against these very fluctuations. This novel ''order from disorder'' phenomenon and the universal ''anomalous elasticity,'' namely, length-scale dependent elastic moduli, non-Hookean stress-strain relation, and a universal negative Poisson ratio [5,6,11,12], are now known to be quite commonly exhibited by many other ''soft'' systems subjected to fluctuations.

In this paper we introduce and explore a universality class of *solid* membranes, which *spontaneously* develop an *in-plane* orientational nematic order. Our motivation is twofold. First, our interest is driven by experimental progress in the synthesis of nematic liquid-crystal elastomers [13], statistically isotropic and homogeneous gels of crosslinked polymers (rubber), with main- or side-chain mesogens, that can spontaneously develop a nematic orientational order. Even in the absence of fluctuations, they were predicted [14] and later observed to display an array of fascinating phenomena [15], the most striking of which is the vanishing of stress for a range of strain, applied transversely to the nematic direction. This striking softness is generic, stemming from the spontaneous orientational symmetry breaking by the nematic state [14,16] that ensures the presence of a zero-energy Goldstone mode, corresponding to the observed [17] soft distortion and strain-induced director reorientation. The hidden rotational symmetry also guarantees the vanishing of one of the five elastic constants [16] that usually characterize harmonic deformations of a three-dimensional uniaxial solid [18]. Thermal fluctuations lead to Grinstein-Pelcovits-like [19] renormalization of elastic constants [14] in bulk systems with dimensions below three in pure systems [20,21] and below five when effects of the random network heterogeneity are taken into account [22]. It is, therefore, likely, and indeed we find that the elastic properties of a two-dimensional fluctuating tensionless sheet of such a nematic elastomer differ *qualitatively* from those of the previously studied crystalline membranes [1]. Our aim here is to explore the effects of thermal fluctuations on this universality class of solid membranes.

Our other motivation for exploring the physics of nematic elastomer membranes comes from an earlier discovery by

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Toner and one of us [23] that any amount of any kind of in-plane anisotropy, a seemingly innocuous generalization, significantly enriches the phase diagram of polymerized membranes. Most dramatically, it was predicted [23] that an entire new phase of membranes, called the ''tubule'' [24] phase, always intervenes between the high-temperature crumpled and low-temperature flat phases. The defining property of the tubule phase is that it is crumpled in one of the two membrane directions but flat (i.e., extended) in the other, with its thickness, roughness, and anomalous elasticity displaying universal behaviors controlled by a nontrivial, infrared stable fixed point [23]. The tubule phase has since been observed in Monte Carlo simulations [27] of non-self-avoiding (i.e., phantom) anisotropic membranes with properties (the thickness, roughness, and Poisson ratio) in close qualitative and quantitative accord with the predictions of Radzihovsky and Toner [23].

Although it was an *explicit* rotational symmetry breaking (anisotropy) that was considered by the authors of Ref. [23], we expect a similar, but qualitatively distinct tubule phase to be also displayed by the spontaneously anisotropic elastomer membranes considered in this paper. However, we leave the subject of the global phase diagram, the elastomer tubule phase, and the phase transitions between it and the flat and crumpled phases for a future publication [28]. Here, we instead focus on the formulation of models of nematically ordered elastomer membranes and the study of long-length-scale properties of its flat phase.

Past extensive investigations, both in the context of defected crystalline membranes [29,30] and liquid crystals confined in rigid gels (e.g., aerogels and aerosils) [31], demonstrated that arbitrarily weak heterogeneity qualitatively modifies the long-scale nature of liquid-crystal orders, membranes morphologies, and elasticities. We, therefore, expect that the local heterogeneity in an elastomer network will also become qualitatively important on sufficiently long scales and will likely dominate over the thermal fluctuation effects that are the subject of this paper. However, here we consider the idealized limit of elastomer membranes in which the effective quenched disorder is sufficiently weak such that its effects are unimportant on scales shorter than some very long disorder-determined length scales. We leave the study of these heterogeneity effects for a future investigation [32].

This paper is organized as follows. In Sec. II, we develop two models for a D-dimensional nematically ordered flat phase of elastomer membranes fluctuating in d dimensions with $d>D$, one with uniaxial nematic order and the other with D-axial nematic order. They are referred to as the uniaxial model and the D-axial model, respectively, in this paper. For the physical case of interest $D=2$, they reduce to the same model, which we study in more detail in Sec. IV. In Sec. III, we investigate harmonic fluctuations of both models and show, in particular, that in-plane phonon fluctuations in $D\geqslant 3$ have parts that are equivalent to those of smectic liquid crystal, a columnar liquid crystal, and a crystalline solid. In Sec. IV, we investigate the mean-field theory and harmonic fluctuations in physically realizable two-dimensional membranes embedded in a three-dimensional space. We find that harmonic fluctuations in these membranes do not destroy the long-range in-plane nematic order in spite of enhanced fluctuations relative to an isotropic system arising from the soft Goldstone mode. In Sec. V, we consider the effects of anharmonicities and develop an $\epsilon=(4-D)$ expansion about the upper critical dimension $D_c=4$ for two model systems that have well-defined analytic continuations to $D=2$. In both models, we find similar flow equation for coupling constants and an infrared-stable fixed point in which the bending modulus κ is unrenmormalized and the in-plane elastic moduli λ_{ij} vanishes with wave number q as q^{ϵ}.

II. MODELS

A. Notation: Reference and target spaces

Although physical membranes are two-dimensional manifolds in a three-dimensional space, it is often useful to consider generalizations to D-dimensional manifolds embedded in a d-dimensional space with $d>D$. Our interest is in solid membranes, which, unlike fluid membranes, have a nonvanishing shear modulus at least in their isotropic state.

To describe the geometry and fluctuations of these membranes, we need to introduce a certain amount of notation. First, we define the reference space S_R. This is the D-dimensional space occupied by the membrane in its quiescent flat reference state. We denote D-dimensional vectors in this space in bold and their components with roman subscripts $i,j=1,\ldots,D$. In particular, we denote intrinsic coordinates on the membrane by the vector

$$\mathbf{r}=(x_1,\ldots,x_D) \tag{2.1}$$

with components x_i. The positions $\mathbf{r}$ index mass points in a Lagrangian description of the membrane. The membrane fluctuates in a d-dimensional target space S_T. We denote vectors in the embedding space with overarrows and components of these vectors with greek subscripts $\mu,\nu=1,\ldots,d$. In particular, we describe the position in S_T of the mass point labeled by $\mathbf{r}$ with the embedding vector

$$\vec{R}(\mathbf{r})=[R_1(\mathbf{r}),\ldots,R_d(\mathbf{r})] \tag{2.2}$$

with components $R_\mu(\mathbf{r})$.

The reference space S_R is identified as a subspace of S_T. The use of orthonormal basis vectors will simplify some of our discussion, and we introduce d vectors $\hat{e}_\mu$ in S_T with components $\hat{e}_{\mu\nu}=\delta_{\mu\nu}$ satisfying

$$\hat{e}_\mu\cdot\hat{e}_\nu=\delta_{\mu\nu}. \tag{2.3}$$

Any vector in S_T can be decomposed into its components along these vectors. For example, $\vec{R}=R_\mu\hat{e}_\mu$, where the summation convention on repeated indices is understood. We choose the set $\{\hat{e}_\mu\}$ so that the subset of vectors $\hat{e}_i$, $i=1,\ldots,D$ lie in S_R, which we represent as a subspace of S_T. Thus,

$$\hat{e}_i\cdot\hat{e}_i=\delta_{ij}, \tag{2.4}$$

$$\hat{e}_\mu \cdot \hat{e}_i = \delta_{\mu i} = \begin{cases} 1 & \text{if } i = \mu \leqslant D \\ 0 & \text{if } \mu = D+1, \ldots, d. \end{cases} \tag{2.5}$$

We choose length scales such that the position in S_T of the point $\mathbf{r}$ in the reference membrane is

$$\vec{R}_{\rm ref} = x_i \hat{e}_i , \tag{2.6}$$

describing a perfectly flat membrane configuration. Positions $\vec{R}(\mathbf{r})$ of the distorted membrane can be described by a D-dimensional in-plane phonon field $\mathbf{u}(\mathbf{r})$ and a $d-D \equiv d_c$-dimensional out-of-plane undulation (height) field $\vec{h}(\mathbf{r})$:

$$\vec{R}(\mathbf{r}) = \sum_{i=1}^{D} [x_i + u_i(\mathbf{r})]\hat{e}_i + \sum_{k=1}^{d_c} h_k(\mathbf{r})\hat{e}_{k+D} . \tag{2.7}$$

Local distortions of the membrane can be expressed in terms of the $d \times D$-dimensional deformation tensor $\underline{\underline{\Lambda}}$ [33] with components

$$\Lambda_{\mu i} = \frac{\partial R_\mu}{\partial x_i} \equiv \partial_i R_\mu . \tag{2.8}$$

In the flat reference state,

$$\Lambda^{\rm ref}_{\mu i} = \frac{\partial R_{{\rm ref},\mu}}{\partial x_i} = \delta_{\mu i} . \tag{2.9}$$

Under independent rotations in the target and reference spaces, $\Lambda_{\mu i}$ transforms as

$$\Lambda_{\mu i} \to \mathcal{O}_{T\mu\nu} \Lambda_{\nu j} \mathcal{O}^{-1}_{Rji} , \tag{2.10}$$

where $\mathcal{O}_{T\mu\nu}$ is a $d \times d$ rotation matrix in S_T and $\mathcal{O}_{Rij}$ is a $D \times D$ rotation matrix in S_R.

Since $\vec{R}$ is a Euclidean vector, the metric tensor, which measures distances between neighboring points via $dR^2 = g_{ij} dx_i dx_j$, for a membrane is

$$g_{ij} = \frac{\partial \vec{R}(\mathbf{r})}{\partial x_i} \cdot \frac{\partial \vec{R}(\mathbf{r})}{\partial x_j} . \tag{2.11}$$

In the reference flat phase, $g_{ij} = g^{\rm ref}_{ij} = \Lambda^{\rm ref}_{\mu i} \Lambda^{\rm ref}_{\mu j} = \delta_{ij}$. We will denote tensors in S_R with a double underscore. Thus, the metric tensor is $\underline{\underline{g}}$. The Lagrangian strain tensor $\underline{\underline{u}}$, with components $u_{ij} = g_{ij} - g^{\rm ref}_{ij}$, measures the local change in separations of nearby points in distorted states characterized by $\vec{R}(\mathbf{r})$ relative to those of the undistorted, flat state characterized by $\vec{R}_{\rm ref}$:

$$dR^2 - dR^2_{\rm ref} = 2 u_{ij} dx_i dx_j . \tag{2.12}$$

Using Eq. (2.7), the strain can be expressed in terms of the phonon and height fields as

$$u_{ij} = \frac{1}{2}(g_{ij} - g^{\rm ref}_{ij}) = \frac{1}{2}\left(\frac{\partial \vec{R}}{\partial x_i} \cdot \frac{\partial \vec{R}}{\partial x_j} - \delta_{ij} \right) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i \mathbf{u} \cdot \partial_j \mathbf{u} + \partial_i \vec{h} \cdot \partial_j \vec{h}). \tag{2.13}$$

B. The model Hamiltonian

As discussed in the Introduction, internally isotropic solid membranes have been extensively studied [1]. In contrast to liquid membranes and one-dimensional polymer analogs, they admit a flat phase, characterized by an infinite orientational (in embedding space) persistence length that is stable to thermal fluctuations. The long-wavelength free-energy density for distortions of such spontaneously flat phase is given by

$$\mathcal{F}_I = \mathcal{F}_{\rm bend} + \mathcal{F}_{\rm stretch} , \tag{2.14}$$

where

$$\mathcal{F}_{\rm bend} = \frac{\kappa}{2} (\nabla^2 \vec{h})^2 \tag{2.15}$$

and

$$\mathcal{F}_{\rm stretch} = \frac{\lambda}{2} ({\rm Tr}\underline{\underline{u}})^2 + \mu ({\rm Tr}\underline{\underline{u}}^2) \tag{2.16a}$$

$$= \frac{B}{2} ({\rm Tr}\underline{\underline{u}})^2 + \mu ({\rm Tr}\underline{\underline{\tilde{u}}}^2), \tag{2.16b}$$

where κ is the membrane's bending rigidity [2], λ and μ are the Lamé coefficients characterizing the in-plane elasticity, $B = \lambda + 2\mu/D$ is the bulk elastic modulus, and

$$\tilde{u}_{ij} = u_{ij} - \frac{1}{D} ({\rm Tr}\underline{\underline{u}})\, \delta_{ij} \tag{2.17}$$

is the symmetric-traceless part of the strain tensor with ${\rm Tr}\underline{\underline{\tilde{u}}} = 0$. Earlier studies of non-liquid-crystalline polymerized membranes [5,6,12] have demonstrated that thermal fluctuations produce wild undulations about the flat state $\vec{R}_{\rm ref}$ that make elastic nonlinearities in the height field $\vec{h}$ (but not in the in-plane phonon fields $\mathbf{u}$) in u_{ij}, Eq. (2.13), important on long length scales and lead to universal length-scale-dependent elastic moduli: $\kappa(\mathbf{q}) \sim q^{-\eta_\kappa}$, $\lambda(\mathbf{q}) \sim q^{\eta_\mu}$, and $\mu(\mathbf{q}) \sim q^{\eta_\mu}$. One of the most important consequences of this is that the thermally driven upward renormalization of the bending rigidity κ stabilizes the low-temperature flat phase of two-dimensional [8] membranes against these very same fluctuations [5,6,12].

The free-energy density of Eq. (2.14) provides a correct description of elastic and height fluctuations of a membrane as long as the equilibrium phase is truly an isotropic flat phase. If, however, the shear modulus μ becomes sufficiently small, the membrane becomes unstable to spontaneous in-plane distortions that break rotational symmetry of S_R [14,16,28]. In the presence of such instability the distortion is stabilized by a nonlinear strain energy $\mathcal{F}_{NL}$ and an in-plane

curvature energy $\mathcal{F}_{\rm curv}$, neglected in $\mathcal{F}_I$, Eqs. (2.14)–(2.16). The in-plane curvature energy, which stabilizes the system against spatially nonuniform distortions is simply

$$\mathcal{F}_{\rm curv}=\tfrac{1}{2}K(\nabla^2\mathbf{u})^2. \tag{2.18}$$

The important low-order contributions to nonlinear strain energy are

$$\mathcal{F}_{NL}=\beta{\rm Tr}\underline{u}{\rm Tr}\underline{\tilde{u}}^2-C{\rm Tr}(\underline{\tilde{u}})^3+\gamma({\rm Tr}\underline{\tilde{u}}^2)^2. \tag{2.19}$$

Terms of order $\underline{u}^5$ or higher and terms proportional to $({\rm Tr}\underline{u})^3$ and $({\rm Tr}\underline{u})^2{\rm Tr}\underline{\tilde{u}}^2$ could also be added but are qualitatively innessential for our present discussion.

C. Spontaneously anisotropic phases

When μ becomes sufficiently small in the presence of $\mathcal{F}_{NL}$, a membrane will undergo a transition [14,16,28] to a new anisotropic, spontaneously stretched equilibrium state with a nonvanishing equilibrium strain $\underline{u}_0$, an anisotropic deformation tensor $\Lambda^0_{\mu i}$ that differs from $\Lambda^{\rm ref}_{\mu i}=\delta_{\mu i}$, and position vectors $R_{0,\mu}=\Lambda^0_{\mu i}x_i$. The most general form of $\Lambda^0_{\mu i}$ can be obtained via target- and reference-space rotations, Eq. (2.10), of the deformation tensor:

$$\Lambda^0_{\mu j}=\begin{cases}\Lambda^0_{ij} & if\ \ \mu=i=1,\ldots,D\\ 0 & \text{if}\ \ \mu=D+1,\ldots,d,\end{cases} \tag{2.20}$$

resulting from a distortion without rotation in the S_R subspace of S_T. Here Λ^0_{ij} are the components of the $D\times D$ matrix $\underline{\Lambda}^0$, restricted to the plane of the original reference space S_R. The new equilibrium $\underline{\Lambda}\Lambda^0$ yields a new metric tensor $g^0_{ij}=\Lambda^0_{\mu i}\Lambda^0_{\mu j}$ and a nonvanishing equilibrium strain relative to the original isotropic flat membrane:

$$u^0_{ij}=\frac{1}{2}(\Lambda^0_{\mu i}\Lambda^0_{\mu j}-\delta_{ij}). \tag{2.21}$$

The simplest anisotropic state that can form is the uniaxial one in which the original isotropic membrane is stretched (compressed) along a single direction in S_R and compressed (stretched) along the others. Because the reference state is isotropic, the direction of stretching, specified by a unit vector $\mathbf{n}^0$ with components n^0_i, is arbitrary in the plane S_R, and we could take it without loss of generality to be along one of the basis vectors, say $\hat{e}_1$. The vector $\mathbf{n}^0$ also exists in the target space S_T. It has components n^0_μ that are equal to n^0_i for $\mu=i=1,\ldots,D$ and zero for $\mu>D$. Thus, in the uniaxial case,

$$\Lambda^0_{ij}=(\Lambda_{0||}-\Lambda_{0\perp})n_{0i}n_{0j}+\Lambda_{0\perp}\delta_{ij}. \tag{2.22}$$

Under rotations in S_T and S_R, this becomes

$$\Lambda^0_{\mu i}=(\Lambda_{0||}-\Lambda_{0\perp})n^T_\mu n^R_i+\Lambda_{0\perp}\hat{e}^T_\mu\cdot\hat{e}^R_i, \tag{2.23}$$

where we used $\delta_{ij}=\hat{e}_{\nu i}\hat{e}_{\nu j}$ and where $n^T_\mu=\mathcal{O}_{T\mu\nu}n_{0\nu}$, $\hat{e}^T_{\nu\mu}=\mathcal{O}_{T\mu i}\hat{e}_{\nu i}$, $\hat{e}^R_{\nu i}=\mathcal{O}^R_{ij}\hat{e}_{\nu j}$, and $n^R_i=\mathcal{O}^R_{ij}n_{0j}$.

The opposite limit of the uniaxial anisotropic state is the full D-axial state in which there is unequal stretching in all D directions in the plane. In this case,

$$\Lambda^0_{ij}=\sum_{k=1}^{D}\Lambda_{0k}e_{ki}e_{kj} \tag{2.24}$$

with all Λ_{0k} different.

As discussed in detail in Ref. [16], the spontaneous broken symmetry described by the above deformation tensors leads to the vanishing of certain shear moduli of the distorted solid. Distortions relative to the new anisotropic state are measured by the displacement and height vectors, $\mathbf{u}'(\mathbf{r}')$ and $\vec{h}(\mathbf{r}')$, defined via

$$\vec{R}'(\mathbf{r}')=\mathbf{r}'+\mathbf{u}'(\mathbf{r}')+\vec{h}(\mathbf{r}')\equiv\vec{R}(\mathbf{r}), \tag{2.25}$$

where by definition $\mathbf{r}'=\vec{R}_0$, $\mathbf{u}'(\mathbf{r}')$ is the D-dimensional in-plane displacement vector, and $\vec{h}(\mathbf{r})$ is the d_c-dimensional ($d_c=D-d$) height variable orthogonal to $\mathbf{u}'$. The strain tensor relative to the new state,

$$u'_{ij}=\frac{1}{2}\left(\frac{\partial R_\mu}{\partial x'_i}\frac{\partial R_\mu}{\partial x'_j}-\delta_{ij}\right), \tag{2.26}$$

expressed in terms of the strain tensor $\underline{u}$ relative to the original state is

$$\underline{u}=\underline{u}_0+\underline{\Lambda}^T_0\underline{\underline{u}'}\underline{\Lambda}_0. \tag{2.27}$$

The strain Hamiltonian density to harmonic order for the uniaxial case

$$\mathcal{F}^{\rm uni}_{\rm strain}=\tfrac{1}{2}\sum_{ij}\lambda^{\rm uni}_{ij}u'_{ii}u'_{jj}+\mu_\perp u'^{\perp}_{ij}u'^{\perp}_{ij}, \tag{2.28}$$

where the Einstein convention is not used in the first term on the right hand side (but is in the second),

$$u'^{\perp}_{ij}=\delta^T_{ik}u'_{kl}\delta^T_{lj}, \tag{2.29}$$

$$\delta^T_{ij}=\delta_{ij}-n_{0i}n_{0j}, \tag{2.30}$$

and

$$\lambda^{\rm uni}_{ij}=\lambda_1\delta_{i1}\delta_{j1}+\lambda_3(1-\delta_{i1})(1-\delta_{j1})+\lambda_2[\delta_{i1}(1-\delta_{j1})+\delta_{j1}(1-\delta_{i1})], \tag{2.31}$$

where the values of λ_1, λ_2, and λ_3 depend on the potentials of the original Hamiltonian and $\underline{\underline{\Lambda}}_0$. This form makes it clear that λ_{ij} is actually a subset of a fourth-rank tensor and not a true second-rank tensor. The bending and curvature energies become anisotropic with

$$\mathcal{F}_{\rm bend}=\tfrac{1}{2}\kappa_{ij}\partial'^2_i\vec{h}\cdot\partial'^2_j\vec{h}, \tag{2.32}$$

$$\mathcal{F}_{\rm curv}=\tfrac{1}{2}K_{ij}\partial'^2_i\mathbf{u}'\cdot\partial'^2_j\mathbf{u}', \tag{2.33}$$

where the anisotropic bending and curvature moduli can be expressed, respectively, as $\kappa_{ij}=\kappa A_{ij}^{\rm uni}$, and $K_{ij}=KA_{ij}^{\rm uni}$, where

$$A_{ij}^{\rm uni}=\Lambda_{0||}^4\delta_{i1}\delta_{j1}+\Lambda_{0\perp}^4(1-\delta_{i1})(1-\delta_{j1})+\Lambda_{0||}^2\Lambda_{0\perp}^2[\,\delta_{i1}(1-\delta_{j1})+\delta_{j1}(1-\delta_{i1})]. \tag{2.34}$$

In the full D-axial case, there is no residual plane that can support shears, and the harmonic free-energy density depends only on the diagonal parts of the strain:

$$\mathcal{F}_{\rm strain}^{\rm D\text{-}axial}=\tfrac{1}{2}\sum_{ij}\lambda_{ij}u'_{ii}u'_{jj}, \tag{2.35}$$

with λ_{ij} being a symmetric matrix with all $D(D+1)/2$ independent entries different. Again, the Einstein convention is suspended in this equation. The bend and curvature energies have the same form as Eqs. (2.33), but $A_{ij}^{\rm uni}$ is replaced by

$$A_{ij}^{\rm D\text{-}axial}=\Lambda_{0i}^2\Lambda_{0j}^2. \tag{2.36}$$

We note that in the most general case that is compatible with the D-axial symmetry, all $D(D+1)/2$ components of $A_{ij}^{\rm D\text{-}axial}$ are independent of each other.

The total Hamiltonian density for a nematic elastomeric membrane is

$$\mathcal{F}=\mathcal{F}_{\rm strain}+\mathcal{F}_{\rm bend}+\mathcal{F}_{\rm curv}. \tag{2.37}$$

We have expressed $\mathcal{F}_{\rm strain}$ only to harmonic order in the nonlinear strain u_{ij} even though nonlinear terms in the original isotropic energy are essential to stabilize the system after the nematic order develops [16]. As we shall see in Sec. V, near four dimensions, anharmonic terms in the nonlinear strains associated with in-plane phonons are subdominant [irrelevant in the renormalization-group (RG) sense] to nonlinearities in the height undulations $\vec{h}$ and we will ignore them in what follows. This is in strong contrast to bulk nematic elastomers, where such phonon nonlinearities cannot be ignored and must be treated nonperturbatively [20,21].

In two dimensions, which we consider in more detail in the Sec. IV, there is only one direction, say the y direction, perpendicular to the direction of order, and $u'^{\perp}_{ij}=\delta_{iy}\delta_{jy}u'_{yy}$. Thus, the second term in Eq. (2.28) can be absorbed into the first term, which depends only on diagonal components of u_{ij}. Thus, in two-dimensional (2D), the free energy density has the same form as Eq. (2.35), and is most naturally analytically continued from D-axial generalization of a nematic elastomer membrane.

III. HARMONIC THEORY

The two (uniaxial and D-axial) nematic membrane Hamiltonians introduced in the preceding section have complicated anisotropies, associated with spontaneous in-plane nematic order, that lead to the membrane's highly anisotropic conformational correlations. In this section, we will investigate fluctuations in the harmonic approximation to these models. These harmonic results will be necessary in our subsequent discussions in Sec. V on the important effects of nonlinearities in the presence of fluctuations. To simplify the notation, in this section we will drop primes appearing in the strain and phonon fields u'_{ij} and $\mathbf{u}'$ measured relative to the nematic state and denote them by u_{ij} and $\mathbf{u}$, not to be confused with physically distinct unprimed quantities of the preceding section.

The harmonic Hamiltonians for the uniaxial and D-axial models have a longitudinal-strain and height elastic parts describing the cost of simple shear modes:

$$\mathcal{H}_L^0=\frac{1}{2}\int\frac{d^Dq}{(2\pi)^D}\Bigg[\sum_{i,j=1}^{D}[\lambda_{ij}q_iq_j+\delta_{ij}K(\hat{q})q^4]u_iu_j+\kappa(\hat{q})q^4\sum_{i=D+1}^{d}h_ih_i\Bigg], \tag{3.1}$$

where

$$K(\hat{q})=\sum_{ij}K_{ij}\hat{q}_i^2\hat{q}_j^2, \tag{3.2}$$

$$\kappa(\hat{q})=\sum_{ij}\kappa_{ij}\hat{q}_i^2\hat{q}_j^2, \tag{3.3}$$

with $\hat{q}_i=q_i/q$. In uniaxial systems, there is in addition a shear part in the $(D-1)$-dimensional plane perpendicular to the unique direction $\mathbf{n}_0$:

$$\mathcal{H}_T^0=\mu_\perp\int d^Dx u_{ij}^\perp u_{ij}^\perp\rightarrow\frac{1}{2}\mu_\perp\int\frac{d^Dq}{(2\pi)^D}[q_\perp^2u_i^\perp u_i^\perp+q_{\perp i}q_{\perp j}u_i^\perp u_j^\perp], \tag{3.4}$$

where $\mathbf{u}^\perp$ is the part of the vector $\mathbf{u}$ in the plane perpendicular to $\mathbf{n}_0$. $\mathbf{u}^\perp$ has $D-1$ components. Thus in $D=2$, $\mathbf{u}^\perp$ has only one component, say u_y, $u_{ij}^\perp=u_{yy}$ and the two models are equivalent.

In both models, at harmonic level, height and phonon variables decouple and the height correlation function is simply

$$G_{h_ih_j}=\delta_{ij}^P\frac{1}{\kappa(\hat{q})q^4}, \tag{3.5}$$

where δ_{ij}^P is the projection operator onto the d_c-dimensional subspace perpendicular to the reference membrane.

A. Uniaxial case

In the uniaxial case, $\mathbf{u}$ can be decomposed into components along the uniaxial direction $\mathbf{n}_0$ and along directions parallel and transverse to the wave vector $\mathbf{q}_\perp$ in the plane perpendicular to $\mathbf{n}_0$:

$$u_i=u_nn_{0i}+u_L\hat{q}_{\perp i}+u_{ti}, \tag{3.6}$$

where $\mathbf{n}_0 \cdot \mathbf{u}_t = 0$ and $\mathbf{q}_\perp \cdot \mathbf{u}_t = 0$. We will represent this decomposition as $u_i = (u_n, u_l, \mathbf{u}_t)$. The $\mathbf{u}$-correlation function can then be decomposed as

$$G_{ij} = G_{nn} n_{0i} n_{0j} + G_{LL} \hat{q}_{\perp i} \hat{q}_{\perp j} + G_{nL}(n_{0i} \hat{q}_{\perp j} + n_{0j} \hat{q}_{\perp i}) + G_t(\delta_{ij}^T - \hat{q}_{\perp i} \hat{q}_{\perp j}). \tag{3.7}$$

The transverse part of G_{ij} decouples from the others and is easily calculated as

$$G_t = \frac{1}{\mu_\perp q_\perp^2}. \tag{3.8}$$

The Hamiltonian for the u_n and u_L components can be written as

$$\mathcal{H} = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \tilde{u}_a G_{ab}^{-1} \tilde{u}_b, \tag{3.9}$$

where $\tilde{u} = (u_n, u_L)$ and

$$G^{-1} = \begin{pmatrix} \lambda_1 q_{||}^2 + K(\hat{q}) q^4 & \lambda_2 q_{||} q_\perp \\ \lambda_2 q_{||} q_\perp & \overline{\lambda}_3 q_\perp^2 + K(\hat{q}) q^4 \end{pmatrix}, \tag{3.10}$$

where $\overline{\lambda}_3 = \lambda_3 + \mu_\perp$ with Λ_i defined in Eq. (2.31). In the long-wavelength limit, terms of order $q_{||}^4$ and $q_{||}^2 q_\perp^2$ can be neglected relative to $q_{||}^2$, and the nn entry in G^{-1} is effectively $\lambda_1 q_{||}^2 + K_\perp q_\perp^4$, which is the inverse of the propagator for fluctuations in a smectic liquid crystal. Similarly, in the long-wavelength limit, the LL part of G^{-1} is effectively $\overline{\lambda}_3 q_\perp^2 + K_{||} q_{||}^4$, which is the inverse propagator for one component of displacement fluctuations in a columnar liquid crystal. Thus, when $\lambda_2 = 0$, G^{-1} decomposes into two independent parts, one of which is identical to the inverse propagator of a smectic liquid crystal and the other of which is identical to one component of the propagator of a columnar liquid crystal. We will find below that this property survives the turning on of λ_2 though with different coefficients.

Equation (3.10) is easily inverted to yield

$$G = \frac{1}{\Delta} \begin{pmatrix} \overline{\lambda}_3 q_\perp^2 + K(\hat{q}) q^4 & -\lambda_2 q_{||} q_\perp \\ -\lambda_2 q_{||} q_\perp & \lambda_1 q_{||}^2 + K(\hat{q}) q^4 \end{pmatrix}, \tag{3.11}$$

where

$$\Delta = \det G^{-1} = \Delta_{\bar{\lambda}} q_\perp^2 q_{||}^2 + (\lambda_1 q_{||}^2 + \overline{\lambda}_3 q_\perp^2) K(\hat{q}) q^4 + \cdots, \tag{3.12}$$

with $\Delta_{\bar{\lambda}} = \lambda_1 \overline{\lambda}_3 - \lambda_2^2$. It is can be shown (see the Appendix for discussion in two dimensions) that G_{nn} is dominated at small $\mathbf{q}$ by the region with $q_{||} \sim q_\perp^2$ so that

$$G_{nn} = \frac{\overline{\lambda}_3 q_\perp^2 + K(\hat{q}) q^4}{\Delta} \to \frac{1}{\lambda_1' q_{||}^2 + K_\perp q_\perp^4}, \tag{3.13}$$

where $\lambda_1' = \Delta_{\bar{\lambda}} / \overline{\lambda}_3 = \lambda_1 - \lambda_2^2 / \overline{\lambda}_3$, $K_\perp = L(\Lambda_{0\perp})^4$. Similarly, G_{LL} is dominated by $q_\perp \approx q_{||}^2$ at small $\mathbf{q}$ and

$$G_{LL} = \frac{\lambda_1 q_{||}^2 + K(\hat{q}) q^4}{\Delta} \to \frac{1}{\lambda_3' q_\perp^2 + K_{||} q_{||}^4}, \tag{3.14}$$

where $\lambda_3' = \Delta_{\bar{\lambda}} / \lambda_1$, $K_{||} = K(\Lambda_{0||})^4$.

Thus we find that, within the harmonic approximation, the translational lower critical dimension is determined by the dominant smecticlike fluctuations of u_n modes and is equal to $D = 3$. The subdominant columnarlike modes u_L become important only below $D = 5/2$, while the transverse fluctuations $\mathbf{u}_t$ only grow with length scale for $D \leqslant 2$.

B. *D*-axial systems

In D-axial systems, the strain energy is given by Eq. (2.35) with $D(D+1)/2$ independent components for λ_{ij}. The harmonic Hamiltonian is then given by Eq. (3.1) with no transverse contributions. The height fluctuations are the same as in the uniaxial case with the appropriate expression for $\kappa(\hat{q})$. The inverse phonon propagator is

$$G^{-1} = \begin{pmatrix} \lambda_1 q_1^2 + K(\hat{q}) q^4 & \cdots & \lambda_{1D} q_1 q_D \\ \vdots & \ddots & \vdots \\ \lambda_{1D} q_1 q_D & \cdots & \lambda_D q_D^2 + K(\hat{q}) q^4 \end{pmatrix}. \tag{3.15}$$

G can be obtained by inverting G^{-1} using minors: $G_{ij} = (-1)^{i+j} \tilde{\Delta}^{ji} / \Delta$, where $\Delta = \det G^{-1}$ and $\tilde{\Delta}^{ij}$ is the determinant of the minor of G^{-1} obtained by deleting row i and column j. The results for the long-wavelength diagonal and off-diagonal components of G_{ij} are, respectively,

$$G_{ii} \approx \frac{1}{\lambda_{ii}' q_i^2 + K(\hat{q}) q^4}, \tag{3.16a}$$

$$G_{ij} \approx \frac{(-1)^{i+j} \tilde{\Delta}_\lambda^{ji} q_i q_j}{\Delta_\lambda q_i^2 q_j^2 + K(\hat{q}) q^4 (\tilde{\Delta}_\lambda^{ii} q_j^2 + \Delta_\lambda^{jj} q_i^2)}, \tag{3.16b}$$

where, as before, Δ_λ is the determinant of λ, $\tilde{\Delta}_\lambda^{ij}$ is the determinant of the ij minor of λ, and $\lambda_{ii}' = \det \Delta_\lambda / \tilde{\Delta}_\lambda^{ij}$. Thus, the diagonal components G_{ii} have the structure of a smectic propagator, with all terms involving q_i in $K(\hat{q}) q^4$ subdominant in the long-wavelength limit.

IV. TWO-DIMENSIONAL MEMBRANES

As discussed in the Introduction, real membranes are two-dimensional manifolds, with reference-space coordinates $\mathbf{r} = (x_1, x_2)$, fluctuating in a three-dimensional space with coordinates $\vec{R} = (R_1, R_2, R_3)$. Because of their low dimensionality, these membranes have a number of special properties, some of which we will investigate in this section. In particular, we will study the transition from an isotropic to a nematically ordered membrane, which, because of the absence of

a third-order invariant for two-dimensional symmetric-traceless tensors is of second order in mean-field theory. Our treatment will review in a two-dimensional context the mechanism for the emergence of soft elasticity in the nematic phase. We will also consider harmonic fluctuations about the flat nematic state in some detail and show that they lead to short-ranged positional correlations, but allow for a stable long-range orientational (nematic) in-plane order of the membrane.

A. The two-dimensional Hamiltonian

Our starting point is the elastic energy of an isotropic two-dimensional membrane augmented by nonlinear elastic terms necessary to stabilize the nematic state that develops when the shear modulus becomes negative. A simplifying feature special to two dimensions is that the strain tensor $\underline{\underline{u}}$ is a 2×2 matrix with no independent cubic invariant. The corresponding Hamiltonian is

$$\mathcal{F}=\frac{\kappa}{2}(\nabla^2 h)^2+\frac{K}{2}(\nabla^2\mathbf{u})^2+\frac{B}{2}(\mathrm{Tr}\underline{\underline{u}})^2+\mu\mathrm{Tr}\tilde{\underline{\underline{u}}}^2+\beta\mathrm{Tr}\underline{\underline{u}}\,\mathrm{Tr}\tilde{\underline{\underline{u}}}^2+\gamma(\mathrm{Tr}\tilde{\underline{\underline{u}}}^2)^2 \tag{4.1a}$$

$$=\frac{\kappa}{2}(\nabla^2 h)^2+\frac{K}{2}(\nabla^2\mathbf{u})^2+\frac{B}{2}\left(\mathrm{Tr}\underline{\underline{u}}+\frac{\beta}{B}\mathrm{Tr}\tilde{\underline{\underline{u}}}^2\right)^2+\left(\gamma-\frac{\beta^2}{2B}\right)\times\left(\mathrm{Tr}\tilde{\underline{\underline{u}}}^2+\frac{\mu/2}{\gamma-\beta^2/(2B)}\right)^2. \tag{4.1b}$$

To simplify our discussion, we have left out the cubic and quartic terms, $(\mathrm{Tr}\underline{\underline{u}})^3$, $(\mathrm{Tr}\underline{\underline{u}})^4$ and $(\mathrm{Tr}\underline{\underline{u}})^2(\mathrm{Tr}\tilde{\underline{\underline{u}}}^2)$, as well as higher-order nonlinearities in $\underline{\underline{u}}$. The inclusion of these terms, which are quantitatively smaller than the terms we have in the nearly incompressible (large B) limit, will not qualitatively modify our results.

The nonlinear strain tensor u^0_{ij}, Eq. (2.21), associated with the spontaneous deformation in the nematic state can easily be found by minimizing the effective energy, Eq. (4.1). Since $\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2$ is greater than or equal to zero, its value $\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2$ at equilibrium is zero as long as $\mu>0$. In this case, $\mathrm{Tr}\underline{\underline{u}}_0$ is also zero. When $\mu<0$, $\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2$ and $\mathrm{Tr}\underline{\underline{u}}_0$ become nonzero with values

$$\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2=\frac{B|\mu|}{2B\gamma-\beta^2}, \tag{4.2a}$$

$$\mathrm{Tr}\underline{\underline{u}}_0=-\frac{\beta}{B}\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2,$$

$$=-\frac{\beta|\mu|}{2B\gamma-\beta^2}. \tag{4.2b}$$

By choosing axes so that the anisotropic stretching is along the x axis, we can express $\underline{\underline{u}}_0$ and $\tilde{\underline{\underline{u}}}_0$ in terms of Λ_{0x} and Λ_{0y} as

$$\underline{\underline{u}}_0=\frac{1}{2}\begin{pmatrix}\Lambda_{0x}^2-1 & 0\\ 0 & \Lambda_{0y}^2-1\end{pmatrix}, \tag{4.3a}$$

$$\tilde{\underline{\underline{u}}}_0=\frac{1}{4}(\Lambda_{0x}^2-\Lambda_{0y}^2)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{4.3b}$$

From this and

$$\underline{\underline{u}}_0=\frac{1}{2}(\mathrm{Tr}\underline{\underline{u}}_0)\underline{\underline{I}}+\tilde{\underline{\underline{u}}}_0 \tag{4.4a}$$

$$=\frac{1}{2}(\mathrm{Tr}\underline{\underline{u}}_0)\underline{\underline{I}}+\frac{1}{2}\sqrt{2\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{4.4b}$$

we obtain

$$\Lambda_{0x}=(1+\mathrm{Tr}\tilde{\underline{\underline{u}}}_0+\sqrt{2\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2})^{1/2}, \tag{4.5}$$

$$\Lambda_{0y}=(1+\mathrm{Tr}\tilde{\underline{\underline{u}}}_0-\sqrt{2\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2})^{1/2}, \tag{4.6}$$

where $\mathrm{Tr}\tilde{\underline{\underline{u}}}_0$ and $\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2$ are given in Eq. (4.2).

We can now write the free energy $\mathcal{F}$, Eq. (4.1), into a more convenient form,

$$\mathcal{F}=\frac{\kappa}{2}(\nabla^2 h)^2+\frac{K}{2}(\nabla^2\mathbf{u})^2+\frac{B}{2}(\mathrm{Tr}\underline{\underline{u}}-\mathrm{Tr}\underline{\underline{u}}_0)^2+\gamma(\mathrm{Tr}\tilde{\underline{\underline{u}}}^2-\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2)^2+\beta(\mathrm{Tr}\underline{\underline{u}}-\mathrm{Tr}\underline{\underline{u}}_0)(\mathrm{Tr}\tilde{\underline{\underline{u}}}^2-\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2), \tag{4.7}$$

which makes it transparent that $\mathcal{F}$ is minimized by $\underline{\underline{u}}=\underline{\underline{u}}_0$ and that permits a straightforward expansion about the ground state.

Using Eq. (2.27), we can now easily express the free-energy density of Eq. (4.7) in terms of the strain $\underline{\underline{u}}'$ relative to the new stretched equilibrium state. First, we observe

$$\mathrm{Tr}\underline{\underline{u}}-\mathrm{Tr}\underline{\underline{u}}0=\mathrm{Tr}\underline{\underline{\Lambda}}_0^2\underline{\underline{u}}', \tag{4.8a}$$

$$\mathrm{Tr}\tilde{\underline{\underline{u}}}^2-\mathrm{Tr}\tilde{\underline{\underline{u}}}_0^2=2\,\mathrm{Tr}(\tilde{\underline{\underline{u}}}_0\underline{\underline{\Lambda}}_0^2\underline{\underline{u}}')+\mathrm{Tr}(\underline{\underline{\Lambda}}_0^2\underline{\underline{u}}')^2-\tfrac{1}{2}(\mathrm{Tr}\underline{\underline{\Lambda}}_0^2\underline{\underline{u}}')^2$$
$$\approx\tfrac{1}{2}(\Lambda_{ox}^2-\Lambda_{0y}^2)(\Lambda_{0x}^2u'_{xx}-\Lambda_{0y}^2u'_{yy}). \tag{4.8b}$$

The final expression, valid to a linear order in $\underline{\underline{u}}'$, does not depend on u'_{xy}—a property, whose origin is the spontaneous broken rotational symmetry of the nematic phase, that is responsible for the vanishing of the membrane shear modulus. Using Eq. (4.8) in Eq. (4.7), replacing ∂_i by $\partial'_i=\Lambda_{0ij}\partial_j$, and retaining only the dominant terms in $\partial_i\mathbf{u}$, we obtain

$$\mathcal{F}=\tfrac{1}{2}\kappa_{xx}(\partial_x^2h)^2+\tfrac{1}{2}\kappa_{yy}(\partial_y^2h)^2+\kappa_{xy}(\partial_x^2h)(\partial_y^2h)+\tfrac{1}{2}K_y(\partial_y^2u_x)^2+\tfrac{1}{2}K_x(\partial_x^2u_y)^2+\tfrac{1}{2}\lambda_xu_{xx}^2+\tfrac{1}{2}\lambda_yu_{yy}^2+\lambda_{xy}u_{xx}u_{yy}, \tag{4.9}$$

where to streamline our notation we again drop primes by replacing $\underline{\underline{u}}'$ with $\underline{\underline{u}}$ and ∂'_i with ∂_i. The strains are the usual nonlinear strains relative to the new reference state, which to

a linear order are simply $u_{xx}=\partial_x u_x$ and $u_{yy}=\partial_y u_y$. The bare elastic coefficients κ's, $K's$, and λ's are determined by the parameters of the original Hamiltonian and $\underline{\underline{\Lambda}}_0$:

$$\kappa_{xx}=\Lambda_{0x}^4\kappa,$$

$$\kappa_{xy}=\Lambda_{0x}^2\Lambda_{0y}^2\kappa,$$

$$\kappa_{yy}=\Lambda_{0y}^4\kappa,$$

$$K_x=\Lambda_{0y}^4 K,$$

$$K_y=\Lambda_{0x}^4 K,$$

$$\lambda_x=\Lambda_{0x}^4[B+\beta(\Lambda_{0x}^2-\Lambda_{0y}^2)+\tfrac{1}{2}\gamma(\Lambda_{0x}^2-\Lambda_{0y}^2)^2],$$

$$\lambda_y=\Lambda_{0y}^4[B-\beta(\Lambda_{0x}^2-\Lambda_{0y}^2)+\tfrac{1}{2}\gamma(\Lambda_{0x}^2-\Lambda_{0y}^2)^2],$$

$$\lambda_{xy}=\Lambda_{0x}^2\Lambda_{0y}^2[B-\gamma(\Lambda_{0x}^2-\Lambda_{0y}^2)^2]. \tag{4.10}$$

B. Fluctuations and correlations of the Harmonic model

We study fluctuations of nematically ordered elastomer membranes within a harmonic approximation for the physically realizable case of $D=2$ and $d=3$. In this case, the displacement vector $\mathbf{u}$ has two components and the height has a single component h. Within this approximation all correlation functions are related to the harmonic two-point correlation functions [34]

$$G_h^0(\mathbf{r})=\langle h(\mathbf{r})h(0)\rangle_0=\int\frac{d^2q}{(2\pi)^2}e^{i\mathbf{q}\cdot\mathbf{r}}G_h^0(\mathbf{q}), \tag{4.11a}$$

$$G_{ij}^0(\mathbf{r})=\langle u_i(\mathbf{r})u_j(0)\rangle_0=\int\frac{d^2q}{(2\pi)^2}e^{i\mathbf{q}\cdot\mathbf{r}}G_{ij}^0(\mathbf{q}), \tag{4.11b}$$

expressed in terms of corresponding "propagators" $G_h^0(\mathbf{q})$ and $G_{ij}^0(\mathbf{q})$. As usual, the averages are computed using a Boltzmann weight $Z_0^{-1}e^{-\mathcal{H}_0[h,\mathbf{u}]}$ (for convenience using k_BT as the energy unit) by integrating over phonon $\mathbf{u}$ and height undulation h fields, with $Z_0=\int DhD\mathbf{u}e^{-\mathcal{H}_0}$ the partition function and $\mathcal{H}_0$ the harmonic effective Hamiltonian $\mathcal{H}_0=\int d^2x\mathcal{F}_0[h,\mathbf{u}]$ obtained from effective Hamiltonian, $\mathcal{H}=\int d^2x\mathcal{F}[h,\mathbf{u}]$ by neglecting all elastic nonlinearities appearing in Eq. (2.13).

The height propagator $G_h^0(\mathbf{q})$ is given by Eq. (3.5) and the phonon propagator $G_{ij}^0(\mathbf{q})$ by Eq. (3.16) specialized to two dimensions:

$$G_{xx}^0(\mathbf{q})=\frac{1}{\lambda_x' q_x^2+K_y q_y^4}, \tag{4.12a}$$

$$G_{yy}^0(\mathbf{q})=\frac{1}{\lambda_y' q_y^2+K_x q_x^4}, \tag{4.12b}$$

$$G_{xy}^0(\mathbf{q})=-\frac{\lambda_{xy}q_xq_y}{\Delta_\lambda q_x^2q_y^2+K(\hat{q})q^4(\lambda_y q_y^2+\lambda_x q_x^2)}, \tag{4.12c}$$

where

$$\lambda_{x,y}'=\lambda_{x,y}\left(1-\frac{\lambda_{xy}^2}{\lambda_x\lambda_y}\right). \tag{4.13}$$

Stability requires $\lambda_x>0$, $\lambda_y>0$, and $\lambda_x\lambda_y-\lambda_{xy}^2>0$. λ_x' and λ_y' both go to zero linearly in $\lambda_x\lambda_y-\lambda_{xy}^2$. The dominant parts of the u_x and u_y correlation functions are identical in form to the displacement correlation functions of a two-dimensional smectic, which have been extensively studied [35]. Both $\langle u_x^2\rangle$ and $\langle u_y^2\rangle$ diverge with system size:

$$\begin{aligned}\langle u_x^2\rangle&=4\int_{2\pi/L_x}^{\infty}\int_{2\pi/L_y}^{\infty}\frac{dq_xdq_y}{(2\pi)^2}G_{xx}^0(\mathbf{q})\\&=\frac{1}{\lambda_x'}\sqrt{\frac{L_x}{2\pi a_x}}\psi_u(2\pi a_xL_x/L_y^2)\\&=\begin{cases}\dfrac{1}{\sqrt{2}\pi\lambda_x'}\sqrt{\dfrac{L_x}{2\pi a_x}} & \text{if } L_y^2\gg 2\pi a_xL_x\\ \dfrac{1}{(2\pi)^2}\dfrac{1}{\lambda_x'}\dfrac{L_y}{a_x} & \text{if } L_y^2\ll 2\pi a_xL_x,\end{cases}\end{aligned} \tag{4.14}$$

where $\psi_u(z)$ is a crossover function. The expression for $\langle u_y^2\rangle$ is obtained by interchanging x and y in the equation for $\langle u_x^2\rangle$. The anisotropy lengths a_x and a_y are defined as

$$a_x=(K_x/\lambda_x')^{1/2}, \tag{4.15a}$$

$$a_y=(K_y/\lambda_y')^{1/2}. \tag{4.15b}$$

They diverge as the stability limit $\lambda_x\lambda_y=\lambda_{xy}^2$ is approached.

The connected phonon correlation functions at two spatially separated points are

$$C_{ij}^0(\mathbf{r})=\langle[u_i(\mathbf{r})-u_i(0)][u_j(\mathbf{r})-u_j(0)]\rangle_0. \tag{4.16}$$

For an infinite membrane

$$\begin{aligned}C_{xx}^0(\mathbf{r})&=2\int\frac{dq_xdq_y}{(2\pi)^2}G_{xx}^0(\mathbf{q})(1-e^{i\mathbf{q}\cdot\mathbf{r}})\\&=\frac{1}{\lambda_x'}\sqrt{\frac{|x|}{\pi a_x}}e^{-y^2/(4a_x|x|)}+\frac{1}{2\lambda_x'}\frac{|y|}{a_x}\mathrm{erf}\left(\frac{|y|}{2\sqrt{a_x|x|}}\right),\\&\sim\begin{cases}\dfrac{1}{\lambda_x'}\sqrt{\dfrac{|x|}{\pi a_x}} & \text{if } |y|\ll 2\sqrt{a_x|x|}\\ \dfrac{1}{\lambda_x'}\dfrac{|y|}{a_x} & \text{if } |y|\gg 2\sqrt{a_x|x|},\end{cases}\end{aligned} \tag{4.17}$$

where erf(x) is the error function. The strong power-law growth of $G^0_{ii}(\mathbf{r})$ indicates that thermal fluctuations lead to arbitrarily large relative displacements of two points distant on the membrane, again contrasting with the usual logarithmic growth with r in two-dimensional ordered xy-like systems.

The stability of the spontaneous nematic (uniaxial) order can be analyzed by examining the rms fluctuations of the nematic director field $\delta\mathbf{n}(\mathbf{r})$. Since in the nematic state, the director is ''massively'' tied to the *antisymmetric* part of the displacement gradient tensor $\eta_{ij}=\partial_j u_i$ [16], orientational fluctuations can be computed from those of phonon fluctuations. To a linear order in rotations of the director $\mathbf{n}$ away from its preferred orientation along the x axis, $\delta n_y=\theta=\frac{1}{2}(\partial_x u_y-\partial_y u_x)$. The angle correlation function is then

$$G_{\theta\theta}(\mathbf{q})=\tfrac{1}{4}[q_y^2 G^0_{xx}(\mathbf{q})+q_x^2 G^0_{xx}(\mathbf{q})-2q_x q_y G^0_{xy}(\mathbf{q})]. \tag{4.18}$$

It is clear from Eqs. (4.12a)–(4.12c) that within the harmonic approximation in-plane orientational fluctuations are finite.

Turning to membrane out-of-plane fluctuations, we find that at harmonic level local rms undulations $\langle h(\mathbf{r})^2\rangle_0$ behave in the same manner as those of polymerized membranes with

$$\langle h(\mathbf{r})^2\rangle_0 = d_c\int_{2\pi/L_x}\int_{2\pi/L_y}\frac{dq_x dq_y}{2\pi}G^0_h(\mathbf{q}) = \frac{d_c(2\pi L_x)^2}{\kappa}\psi_h\left(\frac{L_x}{L_y}\right), \tag{4.19}$$

where for simplicity we specified the case of isotropic bending rigidity κ and defined scaling function

$$\psi_h(z)=4\int_1^\infty dx\int_z^\infty dy\frac{1}{(x^2+y^2)^2}, \tag{4.20}$$

with crossover property

$$\psi_h(z)\to\begin{cases}2\pi & z\to 0\\ \dfrac{2\pi}{z^2} & z\to\infty.\end{cases} \tag{4.21}$$

As in crystalline membranes, the strong L^2 growth implies instability of the flat phase to thermal fluctuations, as well as the importance of anharmonic elasticities, which (as in polymerized membranes) can stabilize the flat phase. Furthermore, in contrast to polymerized membranes, here the power-law divergent, smecticlike in-plane phonon correlations that we found above, Eq. (4.17), suggest that phonon elastic nonlinearities in Eq. (2.13) may be important as well. We turn to these questions in the next two sections.

V. ANHARMONICITIES AND THE ϵ EXPANSION

As discussed in Sec. II, rotational invariance in the target space requires the elastic free energy to be expressed in terms of nonlinear rather than linear strains. The result is that there are anharmonic couplings in the elastic energy both among phonon fields and between phonon and height fields. The violent power-law fluctuations of these fields, the height field in particular, lead, as in other membranes systems [9–12], to divergences with system size L of perturbations in anharmonic couplings for membranes with spatial dimension D less than the critical value D_c. In this section, we consider the interplay between fluctuations and anharmonic couplings in D-dimensional nematic elastomer membranes embedded in a d-dimensional space and show that the perturbation theory breaks down below $D_c=4$. We then study the anomalous elasticity of these membranes in an ϵ expansion about $D=4$. Our interest is in developing an insight into the properties of real two-dimensional membranes. We, therefore, consider only those models that have a straightforward analytic continuation to $D=2$. The two models we consider are the fully anisotropic D-axial elastomer, which has no surviving shear modulus, and the uniaxial model in which we set the shear modulus $\mu_\perp$, Eq. (2.28), for shears in the plane perpendicular to the nematic direction to zero. These two models are equivalent in the physical limit of $D=2$.

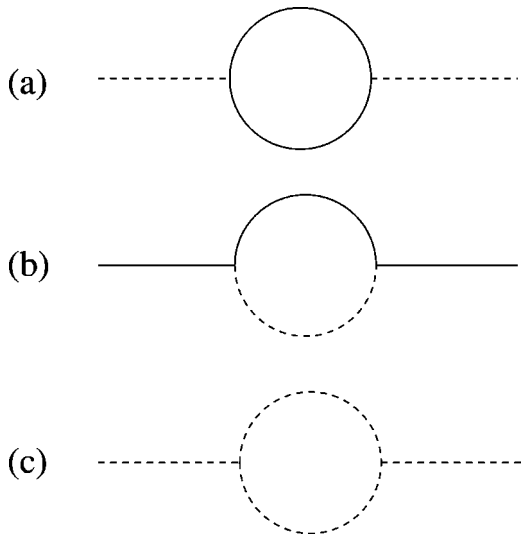

FIG. 1. Feynman diagrams renormalizing elastic constants λ's (a), κ's (b), and K's (c). Solid lines denote undulation fields $\vec{h}$ and dashed lines denote phonon fields $\mathbf{u}$.

A. Perturbative analysis of elastic nonlinearities

The elastic free energy $\mathcal{F}$, Eq. (2.37), contains nonlinearities associated with membrane undulations (involving $\vec{h}$ field) and in-plane phonon nonlinearities. The importance of undulation nonlinearities is a consequence of the membrane's vanishing tension (softness of out-of-plane undulations, controlled by curvature, rather than the surface tension energy). As in an isotropic polymerized membrane [5,6,12], undulation nonlinearities become relevant when the dimension of the reference space is lower than four. To illustrate this point, we calculate the perturbative corrections to elastic constants λ_{ij} from undulation nonlinearities which are represented by Fig. 1(a):

$$\delta\lambda_{ij}^{h}=-\frac{1}{2}\sum_{k,l}^{D}\lambda_{ik}\lambda_{jl}\int\frac{d^{D}q}{(2\pi)^{D}}q_k^2q_l^2[G_h^0(\mathbf{q})]^2$$

$$=-\frac{d_cL^{4-D}}{2(4-D)}\sum_{k,l}^{D}\lambda_{ik}\lambda_{jl}\int\frac{d\Omega_D}{(2\pi)^{D}}\frac{\hat{q}_k^2\hat{q}_l^2}{\kappa^2(\hat{q})}, \quad (5.1)$$

where we assumed that the system has an equilibrium configuration of D-dimensional sphere (disk in two-dimensional case) with radius L. $d\Omega_D$ is the differential surface element of the D-dimensional unit sphere. Quite clearly, for $D<4$, the fluctuation corrections $\delta\lambda_{ij}$ diverge with system size L. The associated nonlinear length scale, beyond which these corrections become comparable to λ_{ij} and the perturbation method breaks down, is given by

$$L_{NL}^{h}\simeq\left(\frac{(4-D)\overline{\kappa^2}}{\lambda}\right)^{1/(4-D)}, \quad (5.2)$$

where λ is the typical value of λ_{ij} and $1/\overline{\kappa^2}$ is defined as the angular integral in Eq. (5.1).

If the modulus $\mu_\perp$ for shears in the direction perpendicular to the anisotropy axis in the uniaxial model is zero, then fluctuations in $\mathbf{u}_t$ Eq. (3.6) have the same q^{-4} harmonic-theory divergence as height fluctuations, but with $K(\hat{q})$ replacing $\kappa(\hat{q})$. Thus, when $\mu_\perp=0$, both $\vec{h}$ and $\mathbf{u}_t$ contribute to divergences in λ_{ij} below $D=4$. This point was missed in the analysis of the fixed-connectivity-fluid fixed point in Ref. [6].

As discussed in the Introduction, the spontaneously broken in-plane rotational symmetry of the nematic elastomer membrane leads to soft in-plane elasticity. As a result, in strong contrast to isotropic or crystalline membranes, in-plane nonlinearities in $\mathcal{F}$, Eq. (2.37), also correct the elastic moduli λ_{ij}. Its contribution $\delta\lambda_{ij}^{u}$, as represented by diagram Fig. 1(c), is given by

$$\delta\lambda_{ij}^{u}=-\frac{1}{2}\sum_{k,l}^{D}\lambda_{ik}\lambda_{jl}\int\frac{d^{D}q}{(2\pi)^{D}}q_k^2q_l^2\sum_{mn}^{D}|G_{mn}^0(\mathbf{q})|^2. \quad (5.3)$$

In the D-axial model, the dominant fluctuations in G_{nm}^0 are smecticlike, and it is straightforward to show that they cause the above correction to λ_{ij}^{u} to diverge as $L^{(3-D)/2}$ below $D=3$ if the phonon fluctuations retained their harmonic character down to $D=3$. In contrast in the uniaxial (analytical continuation) model, the contributions to $\delta\lambda_{ij}^{u}$ due to $\mathbf{u}_t$ fluctuations diverge below $D=4$ when $\mu_\perp=0$, as discussed above. The nn part of G_{nm}^0 is, however, smecticlike and diverges more weakly as $L^{(3-D)/2}$.

A similar calculation shows that the perturbative correction to in-plane elastic constants K_{ij} is dominated by in-plane nonlinearities and also diverges below three dimensions. Thus, the upper critical dimension for undulation nonlinearities is four, while in-plane nonlinearities become relevant below $D=3$. For $3<D<4$, in-plane nonlinearities are irrelevant and, consequently, in-plane curvature energy moduli K_{ij} are only renormalized finitely.

Undulation nonlinearities renormalize bending rigidities κ_{ij} as well as in-plane elastic constants λ_{ij}. One would expect that as in the case of isotropic polymerized membranes, the perturbative corrections to κ_{ij} also diverge as L^{4-D}. This is, however, not true, as one can see from the following argument. Note that if we neglect in-plane nonlinear terms (which is legitimate above three dimensions) in Eq. (2.37) and set K_{ij} to be zero, we can integrate out the phonon fields completely and what is left is only the bending energy $\Sigma_{ij}\kappa_{ij}(\partial_i^2\vec{h}\cdot\partial_j^2\vec{h})$ without any nonlinearities. Thus, there should be no anomalous elasticity for bending rigidity κ_{ij}, if there are no in-plane curvature rigidities K_{ij}. This implies that, in the presence of K_{ij}, renormalization of bending rigidities κ_{ij} is dominated by anisotropic, smecticlike modes, for which in-plane curvature rigidities K_{ij} are important, and the critical dimension below which κ_{ij} is infinitely renormalized is three, as in smectic liquid crystals.

The outcome of the above discussion is that near $D=4$ in the D-axial model, all in-plane nonlinearities in $\mathbf{u}$ are irrelevant and in the uniaxial model nonlinearities in u_n and u_L are irrelevant. Consequently, to capture the long-wavelength behavior we can use simplified expressions for the full nonlinear strain tensors, given by

$$D-\text{axial}: u_{ij}\rightarrow\tfrac{1}{2}(\partial_iu_j+\partial_ju_i+\partial_i\vec{h}\cdot\partial_j\vec{h}), \quad (5.4a)$$

$$\text{uniaxial}: u_{ij}\rightarrow\tfrac{1}{2}(\partial_iu_j+\partial_ju_i+\partial_i\vec{h}\cdot\partial_j\vec{h}+\partial_i\mathbf{u}_t\cdot\partial_j\mathbf{u}_t). \quad (5.4b)$$

The effective Hamiltonian for studying membranes without a shear modulus is $\mathcal{F}$, Eq. (2.37), with $\mu_\perp=0$ and one of the above reduced strains.

The difference between this model free energy and that of isotropic polymerized membranes [5,6], as well as that of fixed-connectivity-fluid membranes [6], should be stressed. First, the bare elastic constants κ_{ij} and λ_{ij} are anisotropic rather than isotropic, and the anisotropy in λ_{ij} cannot be eliminated by a simple rescaling of lengths. Second and more importantly, the energy cost for shear in the planes of anisotropy is *zero* because of the spontaneous broken symmetry of the nematic state. Thus, our model free energy, Eq. (2.37), is not a simple anisotropically scaled generalization of fixed-connectivity fluid [6]. The matrix of coupling constants λ_{ij} is such that the elastic energy cannot be reduced to that of density variation alone.

B. Renormalization group and $(4-D)$ expansion

In isotropic membranes below $D=4$, height fluctuations lead to anomalous elasticity [9–12] with bending modulus and elastic modulus, respectively, diverging and vanishing with wave number as

$$\kappa(\mathbf{q})\sim q^{-\eta_\kappa},\quad \lambda(\mathbf{q})\sim q^{\eta_\lambda}, \quad (5.5)$$

where the exponents η_κ and η_λ are related via the Ward identity [6]

$$2\eta_\kappa+\eta_\lambda=\epsilon=4-D. \quad (5.6)$$

We have just argued that κ is not renormalized for dimensions near four in the models without shear moduli that we are considering. Thus, $\eta_\kappa = 0$ and $\eta_\lambda = \epsilon$. In this section, we will show explicitly within a RG calculation that this is indeed the case in both models we consider.

We use standard momentum-shell renormalization-group procedures, integrating out a shell in momentum space with $\Lambda/e^l < q < \Lambda$, where Λ is the ultraviolet cutoff, to produce fields $\mathbf{u}^<(\mathbf{r})$ and $\vec{h}^<(\mathbf{r})$ with wave numbers with magnitude $q < e^{-l}\Lambda$. We then rescale lengths and fields according to

$$\mathbf{r} = \mathbf{r}' e^l, \tag{5.7}$$

$$u_i^<(\mathbf{r}) = e^{\chi_i l} u_i'(\mathbf{r}'), \tag{5.8}$$

$$\vec{h}^<(\mathbf{r}) = e^{\phi l} \vec{h}'(\mathbf{r}'), \tag{5.9}$$

so as to restore the ultraviolet cutoff to its original value Λ. In the uniaxial case, we decompose $\mathbf{u}$ as in Eq. (3.6) and choose $\chi_n = \chi_L = \chi$ and χ_t different from χ. In the D-axial case, we can choose all the χ_i to be equal to χ. It is convenient to choose the rescaling so as to preserve the nonlinear form of the strain u_{ij}, Eq. (5.4). This requires

$$\chi = 2\phi - 1 = 2\chi_t - 1. \tag{5.10}$$

The integration over the high-wave-vector components of $\mathbf{u}$ and $\vec{h}$ can be performed perturbatively in nonlinearities of Hamiltonian, in a procedure very similar to that of the perturbative analysis. The Feynman graph giving one-loop corrections to λ_{ij} is shown in Fig. 1(a).

After performing the rescaling and calculating the graphic corrections, we obtain the following RG flow equations:

$$\frac{d\lambda_{ij}}{dl} = (D - 2 + 2\chi)\lambda_{ij} - \tfrac{1}{2}\sum_{k,l} \lambda_{ik} M_{kl} \lambda_{lj}, \tag{5.11a}$$

$$\frac{d\kappa_{ij}}{dl} = (D - 4 + 2\phi)\kappa_{ij}, \tag{5.11b}$$

$$\frac{dK_{ij}^n}{dl} = (D - 4 + 2\chi)K_{ij}^n, \tag{5.11c}$$

$$\frac{dK_{ij}^t}{dl} = (D - 4 + 2\chi_t)K_{ij}^t, \tag{5.11d}$$

where, for simplicity, we have set the ultraviolet cutoff $\Lambda = 1$. The components of the in-plane bending coefficient, K_{ij}^n, coupling to u_n and u_L scale with χ, whereas the components K_{ij}^t coupling to $\mathbf{u}_t$ scale with χ_t. The matrix M_{kl} has similar but different forms for the uniaxial and D-axial models. In the D-axial model, $M_{kl} = d_c M_{kl}^\kappa$ and in the uniaxial model $M_{kl} = d_c M_{kl}^\kappa + (D-2) M_{kl}^K$, where

$$M_{kl}^\kappa = \int \frac{d\Omega_D}{(2\pi)^D} \frac{\hat{q}_k^2 \hat{q}_l^2}{\kappa^2(\hat{q})}, \tag{5.12a}$$

$$M_{kl}^K = \int \frac{d\Omega_D}{(2\pi)^D} \frac{\hat{q}_k^2 \hat{q}_l^2}{K^2(\hat{q})}. \tag{5.12b}$$

Note that M_{kl}, like λ_{ij} and A_{ij} defined in Sec. II, is really a subset of the components constructed from the fourth-rank tensor

$$M_{ijkl}^\kappa = \int \frac{d\Omega_D}{(2\pi)^D} \frac{\hat{q}_i \hat{q}_j \hat{q}_k \hat{q}_l}{\kappa^2(\hat{q}) q^4} \tag{5.13}$$

and M_{ijkl}^K defined in a similar way. M_{kl} does not transform like nor have the symmetries of a second-rank tensor. The D-axial and the uniaxial models differ mostly in their respective forms of λ_{ij} and M_{ij}. These different forms require slightly different fixed-point analysis, and we will now treat the two cases separately.

Regardless of the model, we can choose $\phi = (4-D)/2$ and $\chi_t = (4-D)/2$ to keep κ_{ij} and K_{ij}^t fixed. Then, $\chi = 3 - D$ and the inverse correlation function for the parts of $\mathbf{u}$ not in the anisotropy planes (i.e., not $\mathbf{u}_t$) scales as

$$\begin{aligned} G_{ij}^{-1}(\mathbf{q}, \lambda_{ij}, K_{ij}) &= e^{(D-6)l} G_{ij}^{-1}(e^l \mathbf{q}, \lambda_{ij}(l), K_{ij}(l)) \\ &= e^{-\epsilon l} \lambda_{ij}(l) q_i q_j + K(\hat{q}) q^4. \end{aligned} \tag{5.14}$$

Thus, if λ_{ij} has a nonzero fixed point value λ_{ij}^*, then, choosing $e^l q = 1$, we have

$$\lambda_{ij}(q) = \lambda_{ij}^* q^\epsilon. \tag{5.15}$$

Both κ_{ij} and K_{ij}^t remain constant.

C. The uniaxial model

In the uniaxial model, both λ_{ij} Eq. (2.31) and M_{ij} are uniaxial. M_{ij} is easily calculated by taking the appropriate components of the full fourth-rank tensor M_{ijkl}, Eq. (5.13):

$$\begin{aligned} M_{ij} = M_{||}\delta_{i1}\delta_{j1} &+ \frac{1}{(D-1)} M_{||\perp}[\delta_{i1}(1-\delta_{j1}) + (1-\delta_{i1})\delta_{j1}] \\ &+ \frac{1}{(D-1)(D+1)} M_\perp (1+2\delta_{ij})(1-\delta_{i1})(1-\delta_{j1}). \end{aligned} \tag{5.16}$$

The components of M_{ij}^κ and M_{ij}^K of M_{ij} have simple expressions in terms of integrals over angle:

$$M_{||}^\kappa = \int \frac{d\Omega_D}{(2\pi)^D} \frac{\cos^4\theta}{\kappa^2(\hat{q})}, \tag{5.17a}$$

$$M_{||\perp}^\kappa = \int \frac{d\Omega_D}{(2\pi)^D} \frac{\cos^2\theta \sin^2\theta}{\kappa^2(\hat{q})}, \tag{5.17b}$$

$$M_\perp^\kappa = \int \frac{d\Omega_D}{(2\pi)^D} \frac{\sin^4\theta}{\kappa^2(\hat{q})}, \tag{5.17c}$$

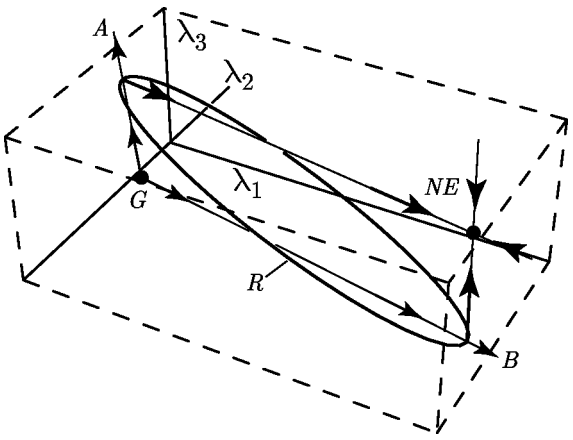

FIG. 2. Renormalization-group flows for the uniaxial model, showing the Gaussian (G) and the uniaxial nematic elastomer (NE) fixed points and the fixed ring R, unstable in one direction and marginally stable around its perimeter. The locus of flow lines, of which GA and GB are two examples, from G to R and extended beyond R form a distorted cone C. All the points within and on the boundary of C flow to the stable fixed-point NE. All other points flow to large coupling.

where θ is the angle $\hat{q}$ makes with the uniaxial axis (one-axis). A similar set of expressions applies to the components of M_{ij}^{K} with $K(\hat{q})$ replacing $\kappa(\hat{q})$. Note that $M_{23}=M_{\perp}/[(D-1)(D+1)]$ and $M_{12}=M_{13}=M_{||\perp}/(D-1)$, whereas the components of the true second-rank uniaxial tensor T_{ij} satisfies $T_{23}=T_{12}=T_{13}=0$.

With these definitions, the recursion relations for the components of λ_{ij} become

$$\frac{d\lambda_1}{dl}=\epsilon\lambda_1-\tfrac{1}{2}(\lambda_1^2M_{||}+2\lambda_1\lambda_2M_{||\perp}+\lambda_2^2M_{\perp}), \tag{5.18a}$$

$$\frac{d\lambda_2}{dl}=\epsilon\lambda_2-\tfrac{1}{2}(\lambda_1\lambda_2M_{||}+\lambda_2^2M_{||\perp}+\lambda_1\lambda_3M_{||\perp}+\lambda_2\lambda_3M_{\perp}), \tag{5.18b}$$

$$\frac{d\lambda_3}{dl}=\epsilon\lambda_3-\tfrac{1}{2}(\lambda_3^2M_{\perp}+2\lambda_2\lambda_3M_{||\perp}+\lambda_2^2M_{||}), \tag{5.18c}$$

where λ_1, λ_2, and λ_3 were defined in Eq. (2.31). These equations have an unusual fixed-point structure as shown in Fig. 2. In appropriate two-dimensional planes in the three-dimensional space of λ_1, λ_2, and λ_3, they exhibit the familiar four-fixed-point structure of systems with two coupled potentials in which there is one unstable Gaussian fixed points, one globally stable fixed point, and two fixed points that are stable in one direction and unstable in the other. The full three-dimensional structure is topologically equivalent to what would be obtained if the two-dimensional structure is rotated about the axis connecting the Gaussian and globally stable fixed points and subsequently stretched anisotropically. In this process, the two mixed-stability fixed points become an elliptical ring with stability exponent of zero for displacements along the ring and one positive and one negative exponent for displacements perpendicular to the ring. The fixed points and their stability exponents ω_1, ω_2, and ω_3 are given as follows.

(a) Gaussian:

$$\lambda_1=\lambda_2=\lambda_3=0,$$

$$\omega_1=\omega_2=\omega_3=\epsilon. \tag{5.19}$$

(2) Uniaxial nematic elastomer:

$$\lambda_1=\frac{2\epsilon M_{\perp}}{\Delta_M},\quad \lambda_2=-\frac{2\epsilon M_{||\perp}}{\Delta_M},\quad \lambda_3=\frac{2\epsilon M_{||}}{\Delta_M},$$

$$\omega_1=-\epsilon,\quad \omega_2=-\epsilon,\quad \omega_3=-\epsilon, \tag{5.20}$$

where $\Delta_M=M_{||}M_{\perp}-M_{||\perp}^2$.

(3) Fixed ring:

$$\lambda_1=\frac{2\epsilon}{M_{||}+2\alpha M_{||\perp}+\alpha^2M_{\perp}},$$

$$\lambda_2=\alpha\lambda_1,\quad \lambda_3=\alpha^2\lambda_1,$$

$$\omega_1=-\epsilon,\quad \omega_2=0,\quad \omega_3=\epsilon \tag{5.21}$$

for $-\infty\leqslant\alpha<\infty$. For every point on the ring, $\lambda_1\lambda_3=\lambda_2^2$. The fixed ring includes the following various interesting points: (a) $\alpha=1$, $\lambda_1=\lambda_2=\lambda_3$, this is the fully isotropic fixed-connectivity-fluid fixed point of isotropic membranes [6]; (b) $\alpha=-1$, $\lambda_1=-\lambda_2=\lambda_3$; (c) $\alpha=\pm\infty$, $\lambda_1=\lambda_2=0$, $\lambda_3=2\epsilon/M_{\perp}$; and (d) $\alpha=0$, $\lambda_1=2\epsilon/M_{||}$, $\lambda_2=\lambda_3=0$.

D. The *D*-axial model

The analysis of the D-axial model is complicated by the fact that λ_{ij} has a large number $[D(D+2)/2=10$ in $D=4]$ of independent components. To simplify our presentation, we will first consider flows in a restricted subspace in which λ_{ij} is parametrized by only two parameters. Thus, we will first find four fixed points with a particular structure and show that one of them is globally stable and describes D-axial nematic elastomer membrane. We then give a general solution for all fixed points and show that it actually contains both the fixed point, Eq. (5.20), and the fixed ring we found for the uniaxial model. Furthermore, we show that if anisotropy of λ_{ij} is turned on in the $(D-1)$-dimensional plane, the uniaxial fixed point, Eq. (5.20), becomes unstable and the system flows to the globally stable D-axial one.

Our recursion relations are still given by Eq. (5.11d) without K_{ij}. Choosing ϕ to keep κ_{ij} constant, we obtain

$$\frac{d\underline{\underline{\lambda}}}{dl}=\epsilon\underline{\underline{\lambda}}-\tfrac{1}{2}\underline{\underline{\lambda}}\,\underline{\underline{M}}\,\underline{\underline{\lambda}}, \tag{5.22}$$

where $\underline{\underline{\lambda}}$ and $\underline{\underline{M}}$ are, respectively, matrices with entries λ_{ij} and M_{ij}. As in the uniaxial case, this equation should have an isotropic fixed point will all λ_{ij} equal, i.e., with $\underline{\underline{\lambda}}\sim\underline{\underline{1}}$ where $\underline{\underline{1}}$ is the matrix with all entries equal to one. It is also

clear that this equation has a fixed point with $\underline{\underline{\lambda}} \sim \underline{\underline{M}}^{-1}$, provided the symmetries of $\underline{\underline{\lambda}}$ and $\underline{\underline{M}}$ are compatible. In the uniaxial case, $\underline{\underline{M}}$ has more unequal components than $\underline{\underline{\lambda}}$, thus the solution $\underline{\underline{\lambda}} \sim \underline{\underline{M}}^{-1}$ is not permitted there. In the D-axial case, $\underline{\underline{M}}$ has the same number of independent components as a D-axial second-rank tensor and $\underline{\underline{\lambda}} \sim \underline{\underline{M}}^{-1}$ is permitted. We thus begin by seeking fixed points in the 2D subspace defined by

$$\underline{\underline{\lambda}} = \lambda_a \underline{\underline{M}}^{-1} + \lambda_b \underline{\underline{1}}. \tag{5.23}$$

The recursion relations for λ_a and λ_b are

$$\frac{d\lambda_a}{dl} = \epsilon \lambda_a - \frac{1}{2}\lambda_a^2, \tag{5.24a}$$

$$\frac{d\lambda_b}{dl} = \epsilon \lambda_b - \frac{1}{2}\lambda_b \left(2\lambda_a + \lambda_b \sum_{i,j=1,}^{D} M_{ij} \right), \tag{5.24b}$$

where we used $\underline{\underline{1}}\, \underline{\underline{M}}\, \underline{\underline{1}} = \Sigma_{ij}^{D} M_{ij} \underline{\underline{1}}$. The fixed points and their stability exponents for these equations are given as follows.

(1) Gaussian:

$$\lambda_a = \lambda_b = 0, \tag{5.25}$$

$$\omega_a = \omega_b = \epsilon. \tag{5.26}$$

(2) Fixed-connectivity fluid:

$$\lambda_a = 0, \quad \lambda_b = \frac{2\epsilon}{\sum_{ij} M_{ij}}, \tag{5.27}$$

$$\omega_a = \epsilon, \quad \omega_b = -\epsilon. \tag{5.28}$$

This fixed point is in fact the isotropic fixed-connectivity-fluid fixed point found in Ref. [6] for isotropic membranes.

(3) Mixed:

$$\lambda_a = 2\epsilon, \quad \lambda_b = -\frac{2\epsilon}{\sum_{ij} M_{ij}}, \tag{5.29}$$

$$\omega_a = -\epsilon, \quad \omega_b = \epsilon. \tag{5.30}$$

We call this fixed point mixed because its coupling constant matrix $\underline{\underline{\lambda}}$ has components characteristic of both the fixed connectivity fluid and the D-axial membrane fixed points.

(4) D-axial nematic elastomer:

$$\lambda_a = 2\epsilon, \quad \lambda_b = 0, \tag{5.31}$$

$$\omega_a = \omega_b = -\epsilon. \tag{5.32}$$

This fixed point is in fact globally stable, i.e., it is stable in all directions. Plugging $\underline{\underline{\lambda}} = \lambda_a \underline{\underline{M}}^{-1}$ into the recursion relation, Eq. (5.22), and linearizing, we obtain $d\,\delta\underline{\underline{\lambda}}/dl = -\epsilon\,\delta\underline{\underline{\lambda}}$. Thus, the stability exponent is $-\epsilon$ for all directions.

We have thus identified one globally stable fixed point, the nematic elastomer fixed point, and two others. The full fixed-point structure of the flow equation, Eq. (5.22), is actually not difficult to determine. Since $\underline{\underline{M}}$ is generically positive definite, we can define a new (symmetric) coupling constant matrix $\underline{\underline{P}} = \underline{\underline{M}}^{1/2} \underline{\underline{\lambda}}\, \underline{\underline{M}}^{1/2}/(2\epsilon)$, whose flow equation is given by

$$\frac{d}{dl}\underline{\underline{P}} = \underline{\underline{P}} - \underline{\underline{P}}^2. \tag{5.33}$$

One can immediately see that every projection matrix [36] is a fixed point for $\underline{\underline{P}}$ and vice versa. This means that the general solution for the flow equation, Eq. (5.22), is given by

$$\underline{\underline{\lambda}} = 2\epsilon \underline{\underline{M}}^{-1/2} \underline{\underline{P}}\, \underline{\underline{M}}^{-1/2}, \tag{5.34}$$

with $\underline{\underline{P}}$ as an arbitrary projection matrix.

We can classify fixed points, or more generally fixed subspaces, by the dimension D_P of the space that P projects onto. Of course, $\mathrm{Tr}P = D_P$. If the dimension D of the elastic manifold is an integer, then P can project onto all subspaces with dimensions $D_P = 0,1,\ldots,D$. When D is not an integer, the classification is less clear. A convenient set, however, is the set with dimensionalities $D_P = 0,1,\ldots,[D]$, and $D_P = D, D-1, \ldots, D-[D]$, where $[D]$ is the greatest integer less than or equal to D. If $D_P = D$, $\underline{\underline{P}}^D = \underline{\underline{I}}$ is the unit matrix, where the superscript indicates the dimension of P. If $D_P = 1$, $P_{ij}^1 = e_i e_j$ for *any* unit vector $\hat{e}$. A $(D-1)$-dimensional operator $P_{ij}^{(D-1)} = \delta_{ij} - e_i e_j$ can also be constructed from the unit vector $\hat{e}$. Similarly, k- and $(D-k)$-dimensional projection operators can be defined via $P_{ij}^k = \Sigma_{l=1}^k e_{li} e_{lj}$ and $P_{ij}^{(D-k)} = \delta_{ij} - P_{ij}^k$, where $\hat{e}_k \cdot \hat{e}_l = \delta_{kl}$.

To study stability of the flow equation, Eq. (5.33), for P for a given fixed-point projection matrix $P_0^{D_0}$, we express deviations of P from $P_0^{D_0}$ as

$$\delta = \underline{\underline{P}} - \underline{\underline{P}}_0^{D_0} = \underline{\underline{P}}_1 + \underline{\underline{P}}_2 + \underline{\underline{P}}_3, \tag{5.35}$$

where

$$\underline{\underline{P}}_1 = \underline{\underline{P}}_0^{D_0}\, \delta\underline{\underline{P}}\, \underline{\underline{P}}_0^{D_0}, \tag{5.36}$$

$$\underline{\underline{P}}_2 = (\underline{\underline{I}} - \underline{\underline{P}}_0^{D_0})\, \delta\underline{\underline{P}}\, (\underline{\underline{I}} - \underline{\underline{P}}_0^{D_0}), \tag{5.37}$$

$$\underline{\underline{P}}_3 = \underline{\underline{P}}_0^{D_0}\, \delta\underline{\underline{P}}\, (\underline{\underline{I}} - \underline{\underline{P}}_0^{D_0}) + (\underline{\underline{I}} - \underline{\underline{P}}_0^{D_0})\, \delta\underline{\underline{P}}\, \underline{\underline{P}}_0^{D_0}. \tag{5.38}$$

Recall that $\underline{\underline{P}}$ is a D-dimensional symmetric matrix with $D(D+1)/2$ independent components. $\underline{\underline{P}}_1$ is the projection of δP onto the D_0-dimensional subspace defined by $\underline{\underline{P}}_0^{D_0}$, and it has $D_0(D_0+1)/2$ independent components. Similarly, $\underline{\underline{P}}_2$ is the projection of $\delta\underline{\underline{P}}$ onto the $D-D_0$ dimensional subspace defined by $\underline{\underline{I}} - \underline{\underline{P}}_0^{D_0}$ with $(D-D_0)(D-D_0+1)/2$ independent components. Finally, $\underline{\underline{P}}_3$ represents the $D_0(D-D_0)$ components of $\delta\underline{\underline{P}}$, which couple the subspaces defined by $\underline{\underline{P}}_0^{D_0}$ and $\underline{\underline{I}} - \underline{\underline{P}}_0^{D_0}$. To linear order, the flow equations for δP are

$$\frac{d}{d\,l}\underline{\underline{P}}_1 = -\underline{\underline{P}}_1, \tag{5.39}$$

$$\frac{d}{d\,l}\underline{\underline{P}}_2 = \underline{\underline{P}}_2, \tag{5.40}$$

$$\frac{d}{d\,l}\underline{\underline{P}}_3 = 0. \tag{5.41}$$

Thus, $\underline{\underline{P}}_1$ is stable, $\underline{\underline{P}}_2$ is unstable, and $\underline{\underline{P}}_3$ is marginally stable. This means that the D_0-dimensional fixed-point projection matrix is stable with respect to any change in $\underline{\underline{P}}$ in the $D_0(D_0+1)/2$-dimensional space of $D_0\times D_0$ matrices that operate in the space defined by $\underline{\underline{P}}_0^{D_0}$; it is unstable with respect to any changes in P in the $(D-D_0)(D-D_0+1)/2$-dimensional space of $(D-D_0)\times(D-D_0)$ matrices that operate in the space defined by $\underline{\underline{I}}-\underline{\underline{P}}_0^{D_0}$; and it is marginally stable with respect to changes in $\underline{\underline{P}}$ in the $D_0(D-D_0)$-dimensional space of matrices that couple $\underline{\underline{P}}_0^{D_0}$ to $\underline{\underline{\delta}}-\underline{\underline{P}}_0^{D_0}$. These results imply that the set of fixed points defined by all D_0-dimensional projections matrices is a $D_0(D-D_0)$-dimensional surface in the space of all possible symmetric matrices P or, equivalently, in the space of coupling constants $\underline{\underline{\lambda}}$. This space is necessarily compact since the subspaces defined by $\underline{\underline{P}}_0^{D_0}$ are parametrized by unit vectors.

There is only one projection operator with $D_0=D$. This is the operator $\underline{\underline{P}}_0^{D}=\underline{\underline{I}}$ that projects onto the whole space. For this case, $\underline{\underline{P}}_2$ and $\underline{\underline{P}}_3$ are both zero and the fixed point is stable in all directions. It is the globally stable fixed point with $\underline{\underline{\lambda}}=2\epsilon\underline{\underline{M}}^{-1}$, which is identical to the stable D-axial fixed point of the restricted set of class of couplings defined by Eq. (5.23). The other fixed points for the restricted set of couplings must correspond to some $\underline{\underline{P}}_0^{D_0}$ with $D_0<D$. It is straightforward to show that the fixed-connectivity-fluid fixed point corresponds to $P^1_{0ij}=e_ie_j$ with

$$e_i=\frac{\sum_{k=1}^{D} M_{ik}^{1/2}}{\left(\sum_{i,j=1}^{D} M_{ij}\right)^{1/2}}, \tag{5.42}$$

where $M_{ij}^{1/2}$ is the ij component of the matrix $\underline{\underline{M}}^{1/2}$. Thus, this fixed point is actually a single point in a $(D-1)=(3-\epsilon)$-dimensional fixed manifold. Similarly, the mixed fixed point corresponds to $P^{D-1}_{0ij}=\delta_{ij}-e_ie_j$. There are other unstable fixed points for $\underline{\underline{\lambda}}$ not described by the restricted set defined by Eq. (5.23), in particular, those with $D_0=2$ or $D-2$.

We have identified all of the fixed-point manifolds of the D-dimensional coupling matrix $\underline{\underline{\lambda}}$. These must include the fixed points of the uniaxial model discussed in the preceding section. When uniaxial constraints are applied, it is natural to construct unit vectors and projection matrices from the components of $\underline{\underline{M}}$ parallel and perpendicular to the anisotropy axis. It is not difficult to show that the stable uniaxial fixed point corresponds to a two-dimensional projection operator

$$P^2_{0ij}=e_{1i}e_{1j}+e_{2i}e_{2j}, \tag{5.43}$$

where

$$e_{1i}=M_{i1}^{1/2}/M_{||}^{1/2}, \tag{5.44}$$

$$e_{2i}=\sqrt{\frac{M_{||}}{\Delta_M}}\left(\sum_{k=2}^{D} M_{ik}^{1/2}-\frac{M_{||\perp}}{M_{||}}M_{1i}^{1/2}\right). \tag{5.45}$$

Thus, the uniaxial fixed point is a point, satisfying uniaxial constraints, in a $2(D-2)$-dimensional fixed manifold of all possible couplings. A point on the uniaxial fixed ring parametrized by α corresponds to a one-dimensional projection operator $\underline{\underline{P}}_0^1$ defined by the unit vector

$$e_i(\alpha)=\frac{M_{i1}^{1/2}+\alpha\sum_{k=2}^{D} M_{ik}^{1/2}}{(M_{||}+2\alpha M_{||\perp}+\alpha^2 M_{\perp})^{1/2}}. \tag{5.46}$$

The set of vectors $\hat{e}(\alpha)$ defined by all α define a one-dimensional loop in a $(D-1)$-dimensional fixed manifold in the space of all possible D-dimensional couplings.

VI. DISCUSSION AND CONCLUSION

In this paper we studied thermal fluctuation and nonlinear elasticity of nematically ordered elastomer membranes in their flat phase. For the physical case of two-dimensional membranes, we found that at harmonic level in-plane phonon correlations are short ranged but the in-plane orientational order remains long ranged, in spite of violent thermal fluctuations. A generalization of a nematic elastomer membrane to an arbitrary D-dimensional membrane with either uniaxial or D-axial nematic order allowed us to study the effects of out-of-plane undulation and in-plane phonon nonlinearities. We found that undulation nonlinearities are relevant when $D<4$ and dominate over the in-plane nonlinearities that only become important when $D<3$. Focusing on the dominant undulation nonlinearities (and neglecting in-plane nonlinearities), we performed a RG calculation combined with an expansion about $D=4$ and found that for $3<D<4$ bending rigidities are only finitely renormalized, while in-plane elastic moduli become singular functions of a wave vector (i.e., exhibit anomalous elasticity), vanishing with a universal power law. This power law is controlled by an infrared stable fixed point whose stability we analyzed in detail for the uniaxial and D-axial analytic continuations, finding agreement in fixed-point structure. This analysis also allowed us to make contact and recover some of the results previously obtained in the studies of (crystalline) polymerized membranes. In particular, we found that the so-called connected fluid is realized as a fixed point of a nematically ordered elastomer membrane that is unstable to the globally stable nematicelastomer fixed point.

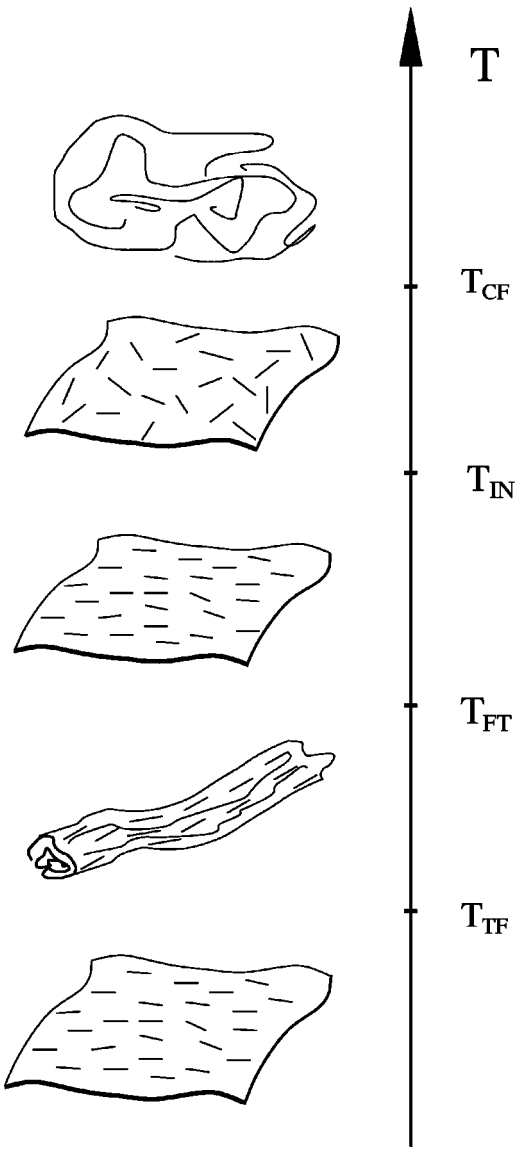

FIG. 3. A possible phase diagram for ideal nematic elastomer membranes. As the temperature is lowered a crumpled membrane undergoes a transition to isotropic flat phase at T_{CF}, followed by a 2D in-plane isotropic-nematic-like transition to an anisotropic(nematic) flat phase. As T is lowered further, this anisotropic flat phase becomes unstable to a nematic tubule phase, where it continuously crumples in one direction but remains extended in the other. At even lower temperature, a tubule-flat transition takes place at T_{TF}.

Despite some of the success in understanding the behavior of nematic elastomer membranes, there are obvious limitations of our analysis, most notably in the application of our work to the physical case of $D=2$ elastomer membranes. This shortcoming primarily has to do with the neglect of in-plane elastic nonlinearities, which near the Gaussian fixed point become relevant for $D<3$. While it is very likely that the subdominance of these in-plane nonlinearities relative to the undulation ones will persist some amount *below* $D=3$ [37], we expect that in the physical case of $D=2$, all the three nonlinearities need to be treated on equal footing. Doing this remains an open and challenging problem.

We conclude with a discussion of the global conformational phase behavior of nematic elastomer membranes. As with ordinary polymerized membranes we expect, upon cooling, isotropic elastomer membranes to undergo a crumpling (flattening) transition from the crumpled to the flat-isotropic phase. Upon further cooling, an in-plane (flat) isotropic to (flat) nematic transition can take place. As shown by Toner and one of us [23], polymerized membranes, with an arbitrary small amount of in-plane anisotropy, inevitably exhibit the so-called tubule phase whose properties and location in the phase diagram are intermediate between the high-temperature crumpled and low-temperature flat phases. Thus, we expect that for nematically ordered elastomer membranes there is a similar nematically ordered tubule phase. Since in such a state the in-plane rotation symmetry is spontaneously (as opposed to explicitly) broken, we expect qualitatively distinct in-plane elasticity distinguished by the presence of a new in-plane soft phonon mode. Consequently, a nematic tubule should be a qualitatively distinct phase of elastic membranes. This discussion is summarized by a possible schematic phase diagram for a nematic elastomer membrane, illustrated in Fig. 3. Also shown in this figure is a possibility of the nematic-flat to nematic-tubule to nematic-flat reentrant phase transitions as a result of competition between growth of nematic order (anisotropy) and suppression of membrane's out-of-plane undulations upon cooling. A detailed analysis of the nematic-tubule phase and these phase transitions will be discussed in a separate publication [28].

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APPENDIX: SCALING OF HARMONIC FLUCTUATIONS AT TWO DIMENSIONS

In this appendix, we show a detailed calculation to justify the approximate form of harmonic phonon propagators in two dimensions, as presented in Eq. (4.12).

The propagators $G^0_{ij}(\mathbf{q})$ are easily found through equipartition or by a Gaussian integration:

$$G^0_{xx}(\mathbf{q})=\frac{1}{\lambda_x q_x^2+K_x q_y^4}\Phi_0\left(\frac{a_x q_y^2}{q_x},\frac{a_y q_x^2}{q_y}\right),$$

$$G^0_{yy}(\mathbf{q})=\frac{1}{\lambda_y q_y^2+K_y q_x^4}\Phi_0\left(\frac{a_x q_y^2}{q_x},\frac{a_y q_x^2}{q_y}\right),$$

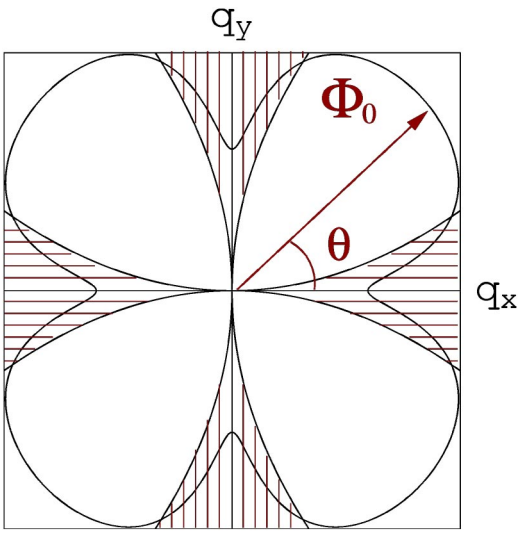

FIG. 4. A polar plot of the crossover function $\Phi_0(a_x q_y^2/q_x, a_y q_x^2/q_y)$, with $q=\sqrt{q_x^2+q_y^2}$ fixed. Shaded are the "decoupled regions" where $|q_x| \ll a_x q_y^2$ or $|q_y| \ll a_y q_x^2$. While Φ_0 is positive and finite for all $\mathbf{q}$, in the limit $q \to 0$ it exhibits cusps at $\theta_n = \arctan(q_y/q_x) = n\pi/2$ $(n=0,1,2,3)$.

$$G^0_{xy}(\mathbf{q}) = -\frac{\lambda_{xy}\phi_0\left(\frac{a_x q_y^2}{q_x}\right)\phi_0\left(\frac{a_y q_x^2}{q_y}\right)}{\lambda_x\lambda_y q_x q_y}\Phi_0\left(\frac{a_x q_y^2}{q_x}, \frac{a_y q_x^2}{q_y}\right),$$
$$= \frac{-\lambda_{xy} q_x q_y \Phi_0\left(\frac{a_x q_y^2}{q_x}, \frac{a_y q_x^2}{q_y}\right)}{(\lambda_x q_x^2 + K_x q_y^4)(\lambda_y q_y^2 + K_y q_x^4)}, \tag{A1}$$

with anisotropy lengths a_x and a_y defined as

$$a_x = (K_x/\lambda_x)^{1/2}, \tag{A2a}$$

$$a_y = (K_y/\lambda_y)^{1/2}. \tag{A2b}$$

Because of the anisotropy of the nematic state the propagators are highly nontrivial even at the harmonic level. Their angular dependence is encoded by the crossover functions

$$\phi_0(z) = \frac{1}{1+z^2}, \tag{A3a}$$

$$\Phi_0(z,w) = \frac{1}{1-\rho^2\phi_0(z)\phi_0(w)}, \tag{A3b}$$

illustrated in Fig. 4 and with asymptotics of the double-crossover function $\Phi_0(z,w)$ given by

$$\Phi_0(z,w) \simeq \begin{cases} \dfrac{1}{1-\rho^2} & \text{if } z \to 0 \text{ and } w \to 0 \\ 1, & \text{if } z \to \infty \text{ or } w \to \infty, \end{cases} \tag{A4}$$

with the ratio

$$\rho^2 = \frac{\lambda_{xy}^2}{\lambda_x\lambda_y}, \tag{A5}$$

required by stability to be less than 1.

As can be seen from the asymptotic form given in Eq. (A4), the scaling function $\Phi_0(a_x q_y^2/q_x, a_y q_x^2/q_y)$ is finite for all $\mathbf{q}$ and therefore simply provides an angular modulation to the phonon correlation functions $G^0_{ij}(\mathbf{q})$. Its value ranges between the "decoupled" and "coupled" values of 1 and $1/(1-\rho^2)$, with the decoupled regime defined by a union of $|q_y| \ll a_y q_x^2$ and $|q_x| \ll a_x q_y^2$ regions in $\mathbf{q}$. The coupled regime is the complement of the decoupled regime $a_x q_y^2 \ll |q_x| \ll a_y^{-1/2}|q_y|^{1/2}$, as illustrated in Fig. 4. As a result, self-correlation (diagonal) functions $G^0_{xx}(\mathbf{q})$ and $G^0_{yy}(\mathbf{q})$ are essentially those of two independent 2D smectics with $x-$ and y-directed layer normals and corresponding phonons u_x and u_y, respectively. The only effect of the cross coupling λ_{xy} on these phonon self-correlation functions is to *finitely* enhance their amplitude in the coupled regime, without modifying their long-wavelength pole structure. Thus, in order to study the fluctuation of u_x field, we only have to concentrate on the wave-vector region $q_x^2 \sim a_x^2 q_y^4$. For $q_x^2 \ll a_x^2 q_y^4$, $\Phi_0(a_x q_y^2/q_x, a_y q_x^2/q_y) \approx 1$ and $G^0_x(\mathbf{q}) \approx 1/K_x q_y^4$, while for $|q_y| \gg q_x^2 \gg a_x^2 q_y^4$, $\Phi_0(a_x q_y^2/q_x, a_y q_x^2/q_y) \approx 1/(1-\rho^2)$ and $G^0_x(\mathbf{q}) \approx 1/\Lambda'_x q_x^2$. This is exactly the same as the asymptotic behaviors of Eq. (4.12a). Thus, we see that Eq. (4.12a) is a good approximation for G^0_x in the wave-vector region $q_x^2 \sim a_x^2 q_y^4$. Of course, for $q_x^4 \gg q_y^2$ the ratio between Eq. (4.12a) and G^0_x is approximately $1/(1-\rho^2)$, but this region is not important for fluctuation of u_x phonon anyway. This analysis also applies to G^0_y if we exchange labels x with y in every place. Thus, Eq. (4.12b) is also a good approximation for G^0_y.

In contrast, the phonon cross-correlation (off-diagonal) function $G^0_{xy}(\mathbf{q})$ depends strongly on whether $\mathbf{q}$ is in the decoupled (union of $|q_y| \ll a_y q_x^2$ and $|q_x| \ll a_x q_y^2$ regions) or coupled $(a_x q_y^2 \ll |q_x| \ll |q_y/a_y|^{1/2})$ regimes. At long scales, near the two dominant x and y smectic regions of $\mathbf{q}$, it is strongly subdominant to the self-correlation functions $G^0_{xx}(\mathbf{q})$ and $G^0_{yy}(\mathbf{q})$, down by a factor of q_y and q_x, respectively.

The subdominance of the cross correlations relative to the self-correlations can also be seen by analyzing the behavior of the cross-correlation ratio of u_x and u_y:

$$\rho_{xy}(\mathbf{q}) = \frac{G^0_{xy}(\mathbf{q})}{\sqrt{G^0_{xx}(\mathbf{q})G^0_{yy}(\mathbf{q})}}. \tag{A6}$$

A simple analysis shows that in the decoupled regime

$$\rho_{xy}(\mathbf{q}) \simeq \begin{cases} \rho \dfrac{|q_x|}{a_x q_y^2} \ll \rho & \text{for } |q_x| \ll a_x q_y^2 \\ \rho \dfrac{|q_y|}{a_y q_x^2} \ll \rho & \text{for } |q_y| \ll a_y q_x^2, \end{cases} \tag{A7}$$

suggesting that thermal fluctuations of u_x and u_y are nearly independent. On the other hand, in the coupled regime

$$\rho_{xy}(\mathbf{q}) \simeq \rho, \quad \text{for } a_x q_y^2 \ll |q_x| \ll |q_y/a_y|^{1/2}, \tag{A8}$$

u_x and u_y are strongly correlated. In this region, we have

$$G_{xy}^0 \approx \frac{\lambda_{xy}}{\lambda_x \lambda_y q_x q_y}. \tag{A9}$$

[1] *Statistical Mechanics of Membranes and Interfaces*, edited by D.R. Nelson, T. Piran, and S. Weinberg (World Scientific, Singapore, 1989).

[2] The surface tension of membranes (which would have contributed to the free energy a term $(\vec{\nabla} h)^2$ that would dominate of the curvature term) identically vanishes in a membrane with unconstrained boundary.

[3] F. Brochard, and J.F. Lennom, J. Phys. (Paris) **36**, 1035 (1975).

[4] W. Helfrich, J. Phys. (Paris) **46**, 1263 (1985); **47**, 322 (1986).

[5] D.R. Nelson and L. Peliti, J. Phys. (Paris) **48**, 1085 (1987).

[6] J.A. Aronovitz and T.C. Lubensky, Phys. Rev. Lett. **60**, 2634 (1988); J.A. Aronovitz, L. Golubovic, and T.C. Lubensky, J. Phys. (Paris) **50**, 609 (1989).

[7] L. Peliti and S. Leibler, Phys. Rev. Lett. **54**, 1690 (1985); D. Foster, Phys. Lett. **114A**, 115 (1986); A.M. Polyakov, Nucl. Phys. B **268**, 406 (1986); F. David, Europhys. Lett. **2**, 577 (1986).

[8] P. Hohenberg, Phys. Rev. **158**, 383 (1967); N.D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).

[9] Y. Kantor, M. Kardar, and D.R. Nelson, Phys. Rev. Lett. **57**, 791 (1986).

[10] M. Paczuski, M. Kardar, and D.R. Nelson, Phys. Rev. Lett. **60**, 2638 (1988).

[11] E. Guitter, F. David, S. Leibler, and L. Peliti, Phys. Rev. Lett. **61**, 2949 (1988).

[12] P. LeDoussal and L. Radzihovsky, Phys. Rev. Lett. **69**, 1209 (1992).

[13] M. Warner and E.M. Terentjev, Prog. Polym. Sci. **21**, 853 (1996), and references therein.

[14] L. Golubovic and T.C. Lubensky, Phys. Rev. Lett. **63**, 1082 (1989).

[15] E.M. Terenjev, J. Phys.: Condens. Matter **11**, R239 (1999).

[16] T.C. Lubensky, R. Mukhopadhyay, L. Radzihovsky, and X. Xing, Phys. Rev. E **66**, 011702 (2002); e-print cond-mat/0112095.

[17] H. Finkelmann, I. Kundler, E.M. Terentjev, and M. Warner, J. Phys. II **7**, 1059 (1997); G.C. Verwey, M. Warner, and E.M. Terentjev, *ibid.* **6**, 1273 (1996); M. Warner, J. Mech. Phys. Solids **47**, 1355 (1999).

[18] L.D. Landau and E.M. Lifshitz, *Theory of Elasticity* (Pergamon, New York, 1970).

[19] G. Grinstein and R. Pelcovits, Phys. Rev. Lett. **47**, 856 (1981).

[20] X. Xing and L. Radzihovsky, Europhys. Lett. (to be published), e-print cond-mat/0212206.

[21] Olaf Stenull and T. C. Lubensky, Europhys. Lett. (to be published), e-print, cond-mat/0212251.

[22] X. Xing and Leo Radzihovsky, e-print cond-mat/0209422.

[23] L. Radzihovsky and J. Toner, Phys. Rev. Lett. **75**, 4752 (1995); Phys. Rev. E **57**, 1832 (1998).

[24] We remind the reader here that this ''tubule'' phase is an anisotropic phase of *polymerized* membranes with a finite in-plane shear rigidity. This should not be confused with a similar, but distinct tubule phase observed in *liquid* lipid membranes studied in Refs. [25,26].

[25] B.N. Thomas, *et al.*, Science **267**, 1635 (1995).

[26] C.-M. Chen, Phys. Rev. E **59**, 6192 (1999).

[27] M. Bowick, M. Falcioni, and G. Thorleifsson, Phys. Rev. Lett. **79**, 885 (1997); e-print cond-mat/9705059.

[28] X. Xing and L. Radzihovsky (unpublished).

[29] L. Radzihovsky and D.R. Nelson, Phys. Rev. A **44**, 3525 (1991).

[30] D.C. Morse, T.C. Lubensky, and G.S. Grest, Phys. Rev. A **45**, R2151 (1992); D.C. Morse and T.C. Lubensky, *ibid.* **46**, 1751 (1992).

[31] L. Radzihovsky and J. Toner, Phys. Rev. B **60**, 206 (1999); Phys. Rev. Lett. **78**, 4414 (1997).

[32] X. Xing, L. Radzihovsky, R. Mukhopadhyay, and T. C. Lubensky (unpublished).

[33] It has to satisfy certain condition of integrability. We will ignore this subtlety in the remaining of this paper.

[34] P. Chaikin and T.C. Lubensky, *Principle of Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).

[35] T. Salditt, I. Koltover, J.O. Rädler, and C.R. Safinya, Phys. Rev. E **58**, 889 (1998).

[36] A projection matrix is a symmetric matrix which equals to the square of itself.

[37] The situation is qualitatively similar to that of the ϕ^4 theory describing Ising transition. There too, the dominant nonlinearity is ϕ^4 that becomes relevant for $D<4$, with the subdominant ϕ^6 nonlinearity near the Gaussian fixed point turning on only for $D \leqslant 3$. Despite this one expects and indeed finds that ϕ^6 operator can be neglected even for $D=3$ and $D=2$, as the relevant analysis needs to be done near the interacting (Wilson-Fisher) and not near the Gaussian critical point. In the same way, we expect that subdominance of in-plane nonlinearities will survive even below $D=3$, but probably not all the way down to the physical case of $D=2$.